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Automated short proof generation for projective geometric theorems with Cayley and bracket algebras

I. Incidence geometry

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Abstract

In this paper we establish the Cayley expansion theory on factored and shortest expansions of typical Cayley expressions in two- and three-dimensional projective geometry. We set up a group of Cayley factorization formulae based on the classification of Cayley expansions. We continue to establish three powerful simplification techniques in bracket computation. On top of the Cayley expansions and simplifications, together with a set of elimination rules, we design an algorithm that can produce extremely short proofs in two- and three-dimensional projective geometry. The techniques developed here can be immediately applied to other symbolic computation tasks involving brackets.

Keywords: Cayley algebra; Bracket algebra; Automated theorem proving; Projective incidence geometry

1. Introduction

Cayley algebra and bracket algebra are an important approach to symbolic computation in applied geometry (Bokowski and Sturmfels, 1989; Crapo and Richter-Gebert, 1994; Hestenes and Ziegler, 1991; McMillan and White, 1991; White, 1975; Whiteley, 1991; Wu, 2001; Li and Wu, 2000a,b; Richter-Gebert, 1996). They are the most general structure

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in which projective properties can be expressed in a coordinate-free way. Because of this nature, they lie closer to synthetic geometry than the usual algebra of coordinates with respect to a fixed frame of reference. In the special issue "Invariant-Theoretic Algorithms in Geometry" of *Journal of Symbolic Computation*, Vol. 11, 1991, there are various applications of the two algebras, particularly in automated geometric theorem proving.

In 1989, Sturmfels and White proved that the algorithm of straightening laws (Doubilet et al., 1974) is a special algorithm to compute Gröbner bases for bracket polynomials, which can be used to prove projective geometric theorems. In 1991 Sturmfels and Whiteley showed that the Cayley factorization techniques (White, 1991) can be used to prove theorems automatically. These are two general coordinate-free approaches to theorem proving with Cayley and bracket algebras.

In addition, there are several other coordinate-free methods for theorem proving using the same algebras, but within a smaller geometric framework. In 1991 Mourrain proposed an elimination method which combines the computation of GCDs of extensors, Cramer's rules and Wu's method. In 1995 Richter-Gebert used the biquadratic final polynomial method (Bokowski and Richter-Gebert, 1990; Sturmfels, 1989) to prove theorems in projective geometry. The proofs are very short in that every polynomial occurring in the proof is composed of two terms.

The straightening algorithm and Cramer's rules are to expand a bracket monomial into a bracket nonmonomial for normalization or elimination. On the other hand, the Cayley factorization is to change a bracket polynomial into a simple Cayley expression. Can we combine these methods so that an automated proof is composed of a minimal number of terms, e.g. two terms at each step? The *import of this topic* lies in that the techniques developed for the purpose of producing short proofs can be immediately used to simplify bracket computing in other applications.

Before answering this question, let us first analyse a general procedure of proving geometric theorems of ruler-constructible type with Cayley and bracket algebras:

Step 1. Cayley algebra representation: represent the hypotheses and conclusion of a theorem by Cayley expressions.

Step 2. Elimination by order: eliminate the geometric entities by their order of construction. This is usually a procedure of substituting the Cayley expressions of the geometric entities into the conclusion, and then expanding the result into a bracket polynomial (*Cayley expansion*).

Step 3. Reduction or factorization: reduce the conclusion, which is now a bracket polynomial, to zero by Grassmann–Plücker (GP) polynomials, or factor the conclusion into a simple Cayley expression.

For a geometric entity or constraint, there are often several different Cayley expressions to represent it. A Cayley expression usually has many ways to be expanded into bracket polynomials. For example, a simple expression like $((\mathbf{1} \vee \mathbf{2}) \wedge (\mathbf{3} \vee \mathbf{4})) \vee ((\mathbf{1}' \vee \mathbf{2}') \wedge (\mathbf{3}' \vee \mathbf{4}')) \vee ((\mathbf{1}'' \vee \mathbf{2}'') \wedge (\mathbf{3}'' \vee \mathbf{4}''))$ in the Cayley algebra of a three-dimensional vector space can have *more than 10 000 expansions* into bracket polynomials (Proposition 3.6 in this paper). These phenomena can make the proving procedures drastically different: while a lucky representation and expansion can lead to an amazingly simple proof, a very unlucky choice can lead to extremely complicated computations.

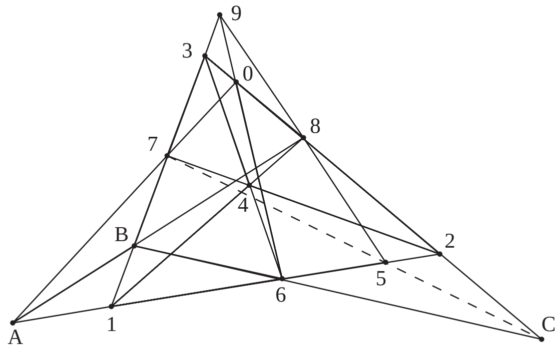

Fig. 1. Example 1.1.

The following is a very lucky and amazing proof, which is discovered by us:

Example 1.1 (Nehring's Theorem, See Chou et al., 1994, Example 6.27). Let $\mathbf{18}$, $\mathbf{27}$, $\mathbf{36}$ be three concurrent lines in triangle $\mathbf{123}$, and let point $\mathbf{5}$ be on side $\mathbf{12}$. Let $\mathbf{9} = \mathbf{13} \cap \mathbf{58}$, $\mathbf{0} = \mathbf{23} \cap \mathbf{69}$, $\mathbf{A} = \mathbf{12} \cap \mathbf{70}$, $\mathbf{B} = \mathbf{13} \cap \mathbf{8A}$, $\mathbf{C} = \mathbf{23} \cap \mathbf{6B}$. Then $\mathbf{5}$, $\mathbf{7}$, $\mathbf{C}$ are collinear.

Free points: **1, 2, 3, 4**.
Semifree point: **5** on **12**.
Intersections:

$$\mathbf{6} = \mathbf{12} \cap \mathbf{34}, \quad \mathbf{7} = \mathbf{13} \cap \mathbf{24}, \quad \mathbf{8} = \mathbf{14} \cap \mathbf{23}, \quad \mathbf{9} = \mathbf{13} \cap \mathbf{58},$$
$$\mathbf{0} = \mathbf{23} \cap \mathbf{69}, \quad \mathbf{A} = \mathbf{12} \cap \mathbf{70}, \quad \mathbf{B} = \mathbf{13} \cap \mathbf{8A}, \quad \mathbf{C} = \mathbf{23} \cap \mathbf{6B}.$$

Conclusion: **5, 7, C** are collinear.

Proof.

Rules

$[\mathbf{67B}]$	$=$	$[\mathbf{136}][\mathbf{78A}]$
$[\mathbf{56B}]$	$=$	$[\mathbf{13A}][\mathbf{568}]$
$[\mathbf{78A}]$	$=$	$-[\mathbf{127}][\mathbf{780}]$
$[\mathbf{13A}]$	$=$	$-[\mathbf{123}][\mathbf{170}]$
$[\mathbf{780}]$	$=$	$-[\mathbf{237}][\mathbf{689}]$
$[\mathbf{170}]$	$=$	$[\mathbf{127}][\mathbf{369}]$
$[\mathbf{689}]$	$=$	$[\mathbf{138}][\mathbf{568}]$
$[\mathbf{369}]$	$=$	$-[\mathbf{136}][\mathbf{358}]$

$$
\begin{aligned}
& [\mathbf{57C}] \\
\overset{\mathbf{C}}{=}\; & -[\mathbf{235}][\mathbf{67B}] - [\mathbf{237}][\mathbf{56B}] \\
\overset{\mathbf{B}}{=}\; & -[\mathbf{136}][\mathbf{235}][\mathbf{78A}] - [\mathbf{13A}][\mathbf{237}][\mathbf{568}] \\
\overset{\mathbf{A}}{=}\; & [\mathbf{127}][\mathbf{136}][\mathbf{235}][\mathbf{780}] + [\mathbf{123}][\mathbf{170}][\mathbf{237}][\mathbf{568}] \\
\overset{\mathbf{0}}{=}\; & \underbrace{[\mathbf{127}][\mathbf{237}]}(-[\mathbf{136}][\mathbf{235}][\mathbf{689}] + [\mathbf{123}][\mathbf{369}][\mathbf{568}]) \\
\overset{\mathbf{9}}{=}\; & \underbrace{[\mathbf{136}][\mathbf{568}]}(-[\mathbf{138}][\mathbf{235}] - [\mathbf{123}][\mathbf{358}]) \\
\overset{\text{contract}}{=}\; & 0.
\end{aligned}
$$

At the beginning of the proof, $[\mathbf{57C}] = (\mathbf{5} \vee \mathbf{7}) \wedge (\mathbf{2} \vee \mathbf{3}) \wedge (\mathbf{6} \vee \mathbf{B})$ has three different expansions into bracket polynomials, the above proof chooses to separate $\mathbf{5}, \mathbf{7}$. In later eliminations by the order of construction, the proof always chooses monomial expansions for such Cayley expressions. Common bracket factors (underbraced) are removed once they are detected. In the last step, by the GP relation $[\mathbf{138}][\mathbf{235}]+[\mathbf{123}][\mathbf{358}] = [\mathbf{135}][\mathbf{238}]$, since $[\mathbf{238}] = 0$ by collinearity, the proof simply evaluates the result to zero. Thus, the proving procedure finishes even before points $\mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}$ are eliminated.

Question 1. How to find monomial expansions for Cayley expression $(\mathbf{1} \vee \mathbf{2}) \wedge (\mathbf{1}' \vee \mathbf{2}') \wedge (\mathbf{1}'' \vee \mathbf{2}'')$?
Answer: By the area method of Chou et al. (1994). This is an elimination method whose rules are derived from properties of signed areas in affine geometry. It can also be used to prove projective geometric theorems.

Question 2. How to find a suitable expansion like the first expansion in the above proof, when there is no monomial expansion?
Answer: In Section 3 of this paper. We have established a theory on factored and shortest expansions of nine typical Cayley expressions in two- and three-dimensional projective geometry.

Question 3. How to use GP relations to reduce the size of a polynomial?
Answer: In Section 5 of this paper. We have developed three powerful simplification techniques in bracket computation, inspired by the Cayley factorization techniques and the straightening algorithm. They can partially solve the problem "computing a bracket representation having the minimal number of tableaux" in Sturmfels (1993, p. 93).

Now we can answer the question raised earlier: we can combine Cayley expansion and Cramer's rules with Cayley factorization and simplification techniques to get *much shorter proofs*. We have developed several powerful Cayley factorization techniques in Section 4, which will be used in the next paper (Li and Wu, 2003) on theorem proving in conic geometry, and from which we have derived a powerful simplification technique called *contraction*. Our central idea to overcome the difficulty of multiple representations, eliminations and expansions is to use "**breefs**"—**b**racket-oriented **r**epresentation, **e**limination and **e**xpansion for **f**actored and **s**hortest results.

Furthermore, in Section 6 of this paper we have developed several elimination rules for theorem proving. They and the Cayley expansion and simplification techniques are put together to form the feature of our short proof generation algorithm in Section 7. Altogether 35 incidence theorems of ruler-constructible type have been tested by the algorithm (Dress and Wenzel, 1991; Graustein, 1930; Hodge and Pedoe, 1953). All of them have 2-termed proofs except one theorem, which is proved not having any 2-termed proof, and for which a very nice 4-termed proof is found. For coordinate-free proving of projective incidence theorems which are not ruler-constructible, some algorithms can be found in Richter-Gebert (1995), Li and Cheng (1997) and Wang (1998), etc.

In the next section we introduce preliminaries of Cayley and bracket algebras and the representation of projective incidence geometry in these algebras.

2. Cayley algebra, brackets and projective geometry

We start with a coordinate-free introduction on Cayley algebra. A thorough exposition of this algebra can be found in Doubilet et al. (1974) and Barnabei et al. (1985). Let $\mathcal{V}$ be an n-dimensional vector space over a field $\mathcal{F}$ whose characteristic is not 2. Then $\mathcal{V}$ generates a *Grassmann algebra* $(\Lambda(\mathcal{V}), \vee)$, where $\Lambda(\mathcal{V})$ is the generated *Grassmann space* and "$\vee$" is the *outer product*. The Grassmann space is graded, whose grades range from 0 to n. Let $\langle x \rangle_r$ denote the r-graded part of $x \in \Lambda(\mathcal{V})$. Let I_n be a fixed nonzero element of grade n in $\Lambda(\mathcal{V})$. The following bilinear form

$$B_{I_n}(x, y) = \langle x \vee y \rangle_n / I_n, \qquad \forall x, y \in \Lambda(\mathcal{V}) \tag{2.1}$$

is nonsingular, and induces a linear invertible mapping $i \colon \Lambda(\mathcal{V}) \longrightarrow \Lambda(\mathcal{V}^*)$, where $\mathcal{V}^*$ is the dual vector space of $\mathcal{V}$. Then $\wedge = i^{-1} \circ \vee \circ i$ defines the *wedge product* in $\Lambda(\mathcal{V})$. The Grassmann space $\Lambda(\mathcal{V})$ equipped with the two products "$\vee$" and "$\wedge$" is called the *Cayley algebra* over $\mathcal{V}$.

The outer product is usually denoted by juxtaposition of elements, and precedes the wedge product by default. Let A_r, B_s be respectively r-graded and s-graded elements of $\Lambda(\mathcal{V})$. Then

$$\begin{aligned} A_r \vee B_s &= (-1)^{rs} B_s \vee A_r \\ A_r \wedge B_s &= (-1)^{(n-r)(n-s)} B_s \wedge A_r. \end{aligned} \tag{2.2}$$

Cayley algebra provides projectively invariant algebraic interpretations of synthetic geometric statements. The following are basics of translating projective geometric incidences into this algebra.

(1) A point is represented by a nonzero vector, which is unique up to scale. It is always denoted by a bold-faced integer or character.

(2) A line passing through points $\mathbf{1}, \mathbf{2}$ is represented by $\mathbf{12}$. A plane passing through points $\mathbf{1}, \mathbf{2}, \mathbf{3}$ is represented by $\mathbf{123}$.

(3) Three points $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are collinear if and only if their outer product is zero. Four points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ are coplanar if and only if their outer product is zero.

(4) Three planar lines $\mathbf{12}, \mathbf{1'2'}, \mathbf{1''2''}$ are concurrent if and only if their wedge product is zero. Four planes $\mathbf{123}, \mathbf{1'2'3'}, \mathbf{1''2''3''}, \mathbf{1'''2'''3'''}$ are copunctual if and only if their wedge product is zero.

Now we introduce bracket algebra, a suitable coordinate-free algebraic setting to deal with projective configurations. For any n-graded element $J_n \in \Lambda(\mathcal{V})$, its *bracket* is defined by

$$[J_n] = J_n / I_n. \tag{2.3}$$

The following is *Cramer's rule* in $\mathcal{V}$: for any $n+1$ vectors $\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}$,

$$[\mathbf{A}_1 \cdots \mathbf{A}_n]\mathbf{B} = \sum_{i=1}^{n} (-1)^{i+1} [\mathbf{B}\mathbf{A}_1 \cdots \check{\mathbf{A}}_i \cdots \mathbf{A}_n] \mathbf{A}_i. \tag{2.4}$$

Here $\check{\mathbf{A}}_i$ denotes the missing of $\mathbf{A}_i$ in the series $\mathbf{A}_1$ to $\mathbf{A}_n$. The following is the *expansion formula* of the wedge product: for vectors $\mathbf{A}_1, \ldots, \mathbf{A}_r$ and $\mathbf{B}_1, \ldots, \mathbf{B}_s$, where $r + s \geq n$,

$$\begin{aligned}
&(\mathbf{A}_1 \cdots \mathbf{A}_r) \wedge (\mathbf{B}_1 \cdots \mathbf{B}_s) \\
&\quad = \sum_{\sigma} \operatorname{sign}(\sigma)[\mathbf{A}_{\sigma(1)} \cdots \mathbf{A}_{\sigma(n-s)} \mathbf{B}_1 \cdots \mathbf{B}_s] \mathbf{A}_{\sigma(n-s+1)} \cdots \mathbf{A}_{\sigma(r)} \\
&\quad = \sum_{\tau} \operatorname{sign}(\tau)[\mathbf{A}_1 \cdots \mathbf{A}_r \mathbf{B}_{\tau(r+s+1-n)} \cdots \mathbf{B}_{\tau(s)}] \mathbf{B}_{\tau(1)} \cdots \mathbf{B}_{\tau(r+s-n)}.
\end{aligned} \tag{2.5}$$

Here σ is a permutation of $1, \ldots, r$ such that $\sigma(1) < \cdots < \sigma(n-s)$ and $\sigma(n-s+1) < \cdots < \sigma(r)$, and τ is a permutation of $1, \ldots, s$ such that $\tau(1) < \cdots < \tau(r+s-n)$ and $\tau(r+s+1-n) < \cdots < \tau(s)$.

Let $\mathbf{A}_1, \ldots, \mathbf{A}_m$ be vectors, and let $[\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_n}]$ be indeterminates over $\mathcal{F}$ for each n-tuple $1 \leq i_1, \ldots, i_n \leq m$, such that they are algebraically independent over $\mathcal{F}$. The *bracket algebra* generated by the $\mathbf{A}$'s over $\mathcal{F}$ is the quotient of the polynomial ring $\mathcal{F}[\{[\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_n}] \mid 1 \leq i_1, \ldots, i_n \leq m\}]$ by the ideal $\mathcal{I}_{n,m}$ generated by elements of the following three types:

B1. $[\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_n}]$ if any $i_j = i_k$, $j \neq k$.

B2. $[\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_n}] - \operatorname{sign}(\sigma)[\mathbf{A}_{i_{\sigma(1)}} \cdots \mathbf{A}_{i_{\sigma(n)}}]$ for any permutation σ of $1, \ldots, n$.

GP (Grassmann–Plücker polynomials). $\sum_{k=1}^{n+1} (-1)^{k+1} [\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_{n-1}} \mathbf{A}_{j_k}][\mathbf{A}_{j_1} \cdots \check{\mathbf{A}}_{j_k} \ldots \mathbf{A}_{j_{n+1}}]$. The set of GP polynomials is denoted by $\mathcal{GP}_{n,m}$.

Let $1 \leq s \leq n$. Let $i_1 < \cdots < i_{s-1}$, $j_1 < \cdots < j_{n+1}$, and $k_1 < \cdots k_{n-s}$ be three subsequences of $1, \ldots, m$. The following is called a *Van der Waerden syzygy*:

$$\begin{aligned}
&\sum_{\sigma} \operatorname{sign}(\sigma)[\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_{s-1}} \mathbf{A}_{j_{\sigma(1)}} \cdots \mathbf{A}_{j_{\sigma(n-s+1)}}] \\
&\quad \times [\mathbf{A}_{j_{\sigma(n-s+2)}} \cdots \mathbf{A}_{j_{\sigma(n+1)}} \mathbf{A}_{k_1} \cdots \mathbf{A}_{k_{n-s}}].
\end{aligned} \tag{2.6}$$

Here σ is a permutation of $1, \ldots, n+1$ such that $\sigma(1) < \cdots < \sigma(n-s+1)$ and $\sigma(n-s+2) < \cdots < \sigma(n+1)$. If furthermore $i_{s-1} < j_{s+1}$ and $j_s < k_1$, then (2.6) is called a *straightening syzygy*. All straightening syzygies form a Gröbner basis of $\mathcal{I}_{n,m}$. The normal form reduction with respect to this basis is called the *straightening algorithm*, see Sturmfels and White (1989) and Sturmfels (1993).

Consider the following problem. Let $\mathbf{A}_1, \ldots, \mathbf{A}_m$ be m vectors in $\mathcal{V}$. For any subsequence $i_1, \ldots, i_r$ of $1, \ldots, m$, vectors $\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_r}$ generate a Grassmann space of maximal grade $s \leq r$. Let I_s be a nonzero s-graded element in this space. Then for any s-graded element J_s in the space, its bracket equals $[J_s]_{I_s} = J_s/I_s$. Different combinations of the $\mathbf{A}$'s generate different Grassmann spaces. For each space we define a bracket. A natural question is how to put these brackets into a single bracket algebra, so that brackets from different spaces can interact with each other?

An elegant solution is provided by Mourrain (unpublished work). Let $[\ ]_n$ be the bracket defined in $\Lambda(\mathcal{V})$. Let $\mathbf{U}_1, \ldots, \mathbf{U}_n$ be n generic vectors in $\mathcal{V}$, called *dummy vectors*. By this we assume that $\mathbf{U}_1 \cdots \mathbf{U}_{n-r}\mathbf{A}_{i_1} \cdots \mathbf{A}_{i_r} \neq 0$ for any subsequence $i_1, \ldots, i_r$ of $1, \ldots, m$. For an r-graded element J_r in $\Lambda(\mathcal{V})$, the following bracket

$$[J_r] = [J_r \mathbf{U}_1 \cdots \mathbf{U}_{n-r}]_n \tag{2.7}$$

is called the *Mourrain bracket* of J_r.

Mourrain brackets unify all brackets defined in the Grassmann subspaces generated by subsets of the $\mathbf{A}$'s. Let I_s be an s-graded element in $\Lambda(\mathcal{V})$ defining a bracket in the corresponding Grassmann subspace, then a new bracket can be defined in the subspace as follows: let $I'_s = I_s/[I_s \mathbf{U}_1 \cdots \mathbf{U}_{n-s}]_n$, then for any s-graded element J_s in the space, define $[J_s]_{I'_s} = J_s/I'_s$. We have

$$[J_s]_{I'_s} = \frac{[J_s \mathbf{U}_1 \cdots \mathbf{U}_{n-s}]_n}{[I'_s \mathbf{U}_1 \cdots \mathbf{U}_{n-s}]_n} = [J_s \mathbf{U}_1 \cdots \mathbf{U}_{n-s}]_n.$$

Because of this unification, hereafter we drop the subscripts of all brackets by assuming that they are Mourrain brackets.

The following are GP relations representable by Mourrain brackets:

(1) Let $1 \leq r \leq s \leq n$, then

$$\sum_{i=1}^{s+1} (-1)^{i+1} [\mathbf{A}_1 \cdots \mathbf{A}_{r-1} \mathbf{B}_i][\mathbf{B}_1 \cdots \check{\mathbf{B}}_i \cdots \mathbf{B}_{s+1}] = 0.$$

(2) Let $1 \leq r \leq n$, then

$$\begin{aligned}\sum_{i=1}^{r} (-1)^{i+1} [\mathbf{A}_1 \cdots \mathbf{A}_{r-1} \mathbf{B}_i][\mathbf{B}_1 \cdots \check{\mathbf{B}}_i \cdots \mathbf{B}_r] \\ = (-1)^{r+1} [\mathbf{B}_1 \cdots \mathbf{B}_r][\mathbf{A}_1 \cdots \mathbf{A}_{r-1}].\end{aligned}$$

3. Cayley expansions in two and three dimensions

In this section, we establish the Cayley expansion theory of nine types of Cayley expressions in two- and three-dimensional projective geometry. The first most important technique is the *computation rules of brackets*, which include (**B1**), (**B2**) in the definition of $\mathcal{I}_{n,m}$, and

(C1) If $\mathbf{1}, \ldots, \mathbf{r}$ are on a line, then for any $1 \leq i, j, k \leq r$, (1) $[\mathbf{ijk}] = 0$, (2) for any point $\mathbf{A}$ in the projective space, $[\mathbf{ijkA}] = 0$.

(C2) If $\mathbf{1}, \ldots, \mathbf{r}$ are in a plane of the projective space, then for any $1 \leq i, j, k, l \leq r$, $[\mathbf{ijkl}] = 0$.

These rules are *automatically applied* whenever a bracket occurs.

3.1. Intersection of two lines

In bracket $[\mathbf{123}]$, if $\mathbf{3} = \mathbf{1'2'} \cap \mathbf{1''2''}$, the bracket equals

$$p_I = \mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{1''2''}. \tag{3.1}$$

Definition 3.1. The *Cayley expansion* of p_I by distributing $\mathbf{1}, \mathbf{2}$ is

$$p_I = [\mathbf{11'2'}][\mathbf{21''2''}] - [\mathbf{21'2'}][\mathbf{11''2''}]. \tag{3.2}$$

There are two other expansions of p_I, which distribute $\mathbf{1'}, \mathbf{2'}$ and $\mathbf{1''}, \mathbf{2''}$ respectively.

Proposition 3.1. *Let $\mathbf{1}, \mathbf{2}, \mathbf{1'}, \mathbf{2'}, \mathbf{1''}, \mathbf{2''}$ be points in the projective plane. Let p_I be defined by* (3.2)*. Assume that when expanding p_I, only the bracket computation rules are applied.*

1. *Zero. An expansion of p_I is zero if and only if one of the following conditions is satisfied:* (1) *one of the pairs, $\mathbf{12}, \mathbf{1'2'}, \mathbf{1''2''}$, is identical points;* (2) *two of the three pairs are collinear points;* (3) *one of the six points is on all the three lines. In the following, assume that $p_I \neq 0$ by expansion.*
2. *Monomial expansion. p_I has a monomial expansion if and only if one of the six points is on two of the three lines $\mathbf{12}, \mathbf{1'2'}, \mathbf{1''2''}$.*
3. *Unique expansion. A* double point *is a point in p_I which occurs twice. p_I has a unique expansion if and only if it has two double points. The expansion is*

$$\mathbf{12} \wedge \mathbf{12'} \wedge \mathbf{22''} = [\mathbf{122'}][\mathbf{122''}]. \tag{3.3}$$

Definition 3.2. The multiplication of two p_I-typed wedge products is

$$p_{II} = (\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56})(\mathbf{1'2'} \wedge \mathbf{3'4'} \wedge \mathbf{5'6'}). \tag{3.4}$$

The *Cayley expansion* of p_{II} by separating $\mathbf{1}, \mathbf{2}$ is expanding p_{II} into a bracket polynomial through

$$p_{II} = [\mathbf{134}][\mathbf{256}]\mathbf{1'2'} \wedge \mathbf{3'4'} \wedge \mathbf{5'6'} - [\mathbf{234}][\mathbf{156}]\mathbf{1'2'} \wedge \mathbf{3'4'} \wedge \mathbf{5'6'}. \tag{3.5}$$

There are six such expansions of p_{II}.

Definition 3.3. Let $p = p_1 + p_2$ be an expansion formula of p into two groups of bracket polynomials p_1, p_2. If for certain p, one monomial of p_1 is identical with another monomial of p_2 up to coefficient, the expansion of the p is called an *expansion with like terms*.

Proposition 3.2. *Let $\mathbf{1}, \ldots, \mathbf{6}, \mathbf{1'}, \ldots, \mathbf{6'}$ be points in the projective plane. Let p_{II} be defined by* (3.4)*. Assume that when expanding p_{II}, only the bracket computation rules are used.*

1. *Zero. An expansion of p_{II} is zero if and only if one of its wedge products is zero. In the following, assume that $p_{II} \neq 0$ by expansion.*
2. *Inner point. If $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are collinear, point $\mathbf{3}$ is called an* inner point *in p_{II}. Then*

$$p_{II} = [\mathbf{124}][\mathbf{356}]\mathbf{1'2'} \wedge \mathbf{3'4'} \wedge \mathbf{5'6'} \tag{3.6}$$

is a shortest expansion. If $\{\mathbf{1}, \ldots, \mathbf{6}, \mathbf{1'}, \ldots, \mathbf{6'}\}$ is a set of generic points, and $\mathbf{3} = \mathbf{1}$ or $\mathbf{2}$, then (3.6) *is the unique shortest expansion.*
3. *Like terms. If p_{II} has no inner point, and the two wedge products are not identical, the following is the only pattern which has an expansion with like terms:*

$$(\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56})(\mathbf{13} \wedge \mathbf{24} \wedge \mathbf{56}) = [\mathbf{124}][\mathbf{134}][\mathbf{256}][\mathbf{356}] - [\mathbf{123}][\mathbf{234}][\mathbf{156}][\mathbf{456}]. \tag{3.7}$$

The expansion with like terms is unique.

4. p_{II} *has monomial expansion if and only if both of its wedge products have monomial expansions. The shortest expansions of* p_{II} *are* 2*-termed if either it is of the pattern* (3.7)*, or a wedge product of* p_{II} *has inner point. The shortest expansions of* p_{II} *are* 3*-termed if the two wedge products are identical. In other cases, the shortest expansions are* 4*-termed.*

Proof. Item 1 is trivial. First we prove item 3.

3. By symmetry we only need to consider the expansion (3.5). Since $p_{II} \neq 0$, none of $[\mathbf{134}]$, $[\mathbf{256}]$ can equal $[\mathbf{234}]$, $[\mathbf{156}]$. If there are any like terms, then $[\mathbf{134}][\mathbf{256}]$ must be in an expansion of $\mathbf{1'2'} \wedge \mathbf{3'4'} \wedge \mathbf{5'6'}$. So $\mathbf{1'}, \dots, \mathbf{6'}$ is a permutation of $\mathbf{1}, \dots, \mathbf{6}$. Below we assume that $[\mathbf{134}], [\mathbf{256}], [\mathbf{234}], [\mathbf{156}] \neq 0$, otherwise there are no like terms.

If $\{\mathbf{12}, \mathbf{34}, \mathbf{56}\} = \{\mathbf{1'2'}, \mathbf{3'4'}, \mathbf{5'6'}\}$, then we can assume that $\mathbf{i'} = \mathbf{i}$ for $1 \leq i \leq 6$. After combining like terms, we get three 3-termed expansions of $(\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56})^2$, which correspond to $([\mathbf{124}][\mathbf{356}] - [\mathbf{123}][\mathbf{456}])^2$ and the squares of two other expansions of $\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56}$.

Assume that $\{\mathbf{12}, \mathbf{34}, \mathbf{56}\} \neq \{\mathbf{1'2'}, \mathbf{3'4'}, \mathbf{5'6'}\}$. Without loss of generality, assume that $\mathbf{134} = \mathbf{1'3'4'}$, $\mathbf{256} = \mathbf{2'5'6'}$ up to the order of points. By symmetry there are only two possibilities: either $\mathbf{1} = \mathbf{1'}, \mathbf{3} = \mathbf{3'}, \mathbf{4} = \mathbf{4'}$ and $\mathbf{2} = \mathbf{5'}, \mathbf{6} = \mathbf{6'}, \mathbf{5} = \mathbf{2'}$, or $\mathbf{1} = \mathbf{3'}, \mathbf{4} = \mathbf{4'}, \mathbf{3} = \mathbf{1'}$ and $\mathbf{2} = \mathbf{5'}, \mathbf{6} = \mathbf{6'}, \mathbf{5} = \mathbf{2'}$. For the like terms to occur, there are four cases, in which $[\mathbf{234}][\mathbf{156}]$ is identical to (1) $[\mathbf{2'3'4'}][\mathbf{1'5'6'}]$, (2) $[\mathbf{1'2'3'}][\mathbf{4'5'6'}]$, (3) $[\mathbf{1'2'5'}][\mathbf{3'4'6'}]$, (4) $[\mathbf{1'2'6'}][\mathbf{3'4'5'}]$, respectively.

Case (1): $[\mathbf{2'3'4'}][\mathbf{1'5'6'}]$ is identical to $[\mathbf{534}][\mathbf{126}]$ or $[\mathbf{514}][\mathbf{326}]$. Only the latter one can be identical to $[\mathbf{234}][\mathbf{156}]$ under the condition $\mathbf{4} = \mathbf{6}$. The pattern is $(\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{54})(\mathbf{35} \wedge \mathbf{14} \wedge \mathbf{24})$. The result after combining like terms is the same with that by expanding both wedge products into $[\mathbf{234}][\mathbf{145}] - [\mathbf{134}][\mathbf{245}]$.

Case (2): $[\mathbf{1'2'3'}][\mathbf{4'5'6'}]$ is identical to $[\mathbf{153}][\mathbf{426}]$, which is identical to $[\mathbf{234}][\mathbf{156}]$ under the condition $\mathbf{3} = \mathbf{6}$. There are two patterns: $(\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{53})(\mathbf{15} \wedge \mathbf{34} \wedge \mathbf{23})$ and $(\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{53})(\mathbf{35} \wedge \mathbf{14} \wedge \mathbf{23})$. The results after cancelling like terms are identical with those by expanding the two wedge products of p_{II} into monomials.

Case (3): $[\mathbf{1'2'5'}][\mathbf{3'4'6'}]$ is identical to $[\mathbf{152}][\mathbf{346}]$ or $[\mathbf{352}][\mathbf{146}]$. The former one cannot be identical to $[\mathbf{234}][\mathbf{156}]$, while the latter one can, if $\mathbf{4} = \mathbf{5}$. The result after cancelling like terms is identical with that by expanding the wedge products into monomials.

Case (4): $[\mathbf{1'2'6'}][\mathbf{3'4'5'}]$ is identical to $[\mathbf{156}][\mathbf{342}]$ or $[\mathbf{356}][\mathbf{142}]$. The latter one cannot be identical to $[\mathbf{234}][\mathbf{156}]$. The former one has the pattern $(\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56})(\mathbf{15} \wedge \mathbf{34} \wedge \mathbf{26})$, which corresponds to (3.7). If it is a monomial expansion, then both wedge products of p_{II} have monomial expansions.

2. First, assume that $\{\mathbf{1}, \dots, \mathbf{6}, \mathbf{1'}, \dots, \mathbf{6'}\}$ is a set of generic points, and $\mathbf{3} = \mathbf{1}$. If p_{II} has a monomial expansion, since the bracket ring is a unique factorization domain, the wedge product in (3.6) must have a monomial expansion. Thus, (3.6) is the shortest. Among the expansions of p_{II} having no like terms, (3.6) is obviously the unique shortest one. By the proof of item 3, (3.6) is the unique shortest expansion in cases (1)–(3) and in the case $\{\mathbf{12}, \mathbf{34}, \mathbf{56}\} = \{\mathbf{1'2'}, \mathbf{3'4'}, \mathbf{5'6'}\}$. In case (4), the expansion of $(\mathbf{12} \wedge \mathbf{14} \wedge \mathbf{56})(\mathbf{15} \wedge \mathbf{14} \wedge \mathbf{26})$ by (3.7) gives the same result as (3.6).

Second, let $\mathbf{1}, \ldots, \mathbf{6}, \mathbf{1}', \ldots, \mathbf{6}'$ be points in the plane. We prove that if the wedge product in (3.6) has no monomial expansion, then any other expansion of p_{II} has at least two terms. The assumption indicates (1) $\mathbf{1}, \mathbf{2}, \mathbf{4}$ are not collinear, (2) $\mathbf{3}, \mathbf{5}, \mathbf{6}$ are not collinear, (3) $\mathbf{1}', \mathbf{2}'$ are not on lines $\mathbf{3}'\mathbf{4}', \mathbf{5}'\mathbf{6}'$, (4) $\mathbf{3}', \mathbf{4}'$ are not on lines $\mathbf{1}'\mathbf{2}', \mathbf{5}'\mathbf{6}'$, (5) $\mathbf{5}', \mathbf{6}'$ are not on lines $\mathbf{1}'\mathbf{2}', \mathbf{3}'\mathbf{4}'$.

When an expansion does not have like terms, the expansion result is obviously longer than the shortest result from by (3.6). By the proof of item 3, in cases (1)–(3) and in the case $\{\mathbf{12}, \mathbf{34}, \mathbf{56}\} = \{\mathbf{1}'\mathbf{2}', \mathbf{3}'\mathbf{4}', \mathbf{5}'\mathbf{6}'\}$, the expansion (3.6) is the shortest. In case (4), since the second wedge product of p_{II} has no monomial expansion, neither does (3.7).

4. The proof is already included in those of items 2 and 3. $\square$

From now on, all the proofs of the theorems on Cayley expansions are omitted.

3.2. Two intersections of lines

In bracket $[\mathbf{123}]$, when $\mathbf{2} = \mathbf{1}'\mathbf{2}' \cap \mathbf{3}'\mathbf{4}'$ and $\mathbf{3} = \mathbf{1}''\mathbf{2}'' \cap \mathbf{3}''\mathbf{4}''$, the bracket equals

$$p_{III} = [\mathbf{1}(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')(\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'')]. \tag{3.8}$$

Definition 3.4. The *Cayley expansion* of p_{III} by distributing the pair of lines $\mathbf{1}'\mathbf{2}', \mathbf{3}'\mathbf{4}'$, is expanding p_{III} into a bracket polynomial through

$$p_{III} = [\mathbf{13}'\mathbf{4}']\mathbf{1}'\mathbf{2}' \wedge \mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'' - [\mathbf{11}'\mathbf{2}']\mathbf{3}'\mathbf{4}' \wedge \mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}''. \tag{3.9}$$

There are two distributive expansions of p_{III}. The *hybrid Cayley expansion* of p_{III} by separating the pair of points $\mathbf{1}', \mathbf{2}'$, is expanding p_{III} into a bracket polynomial through

$$p_{III} = [\mathbf{1}'\mathbf{3}'\mathbf{4}']\mathbf{12}' \wedge \mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'' - [\mathbf{2}'\mathbf{3}'\mathbf{4}']\mathbf{11}' \wedge \mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}''. \tag{3.10}$$

There are four hybrid expansions of p_{III}.

Proposition 3.3. *If all the points in p_{III} are generic ones, then p_{III} has* 46 *different expansions into bracket polynomials.*

Proof. Denote the hybrid expansion separating points $\mathbf{i}, \mathbf{j}$ by $h_{\mathbf{ij}}$. By (3.10), $h_{\mathbf{1}'\mathbf{2}'}$ has $3 \times 3 = 9$ expansions into bracket polynomials, one of which has brackets $[\mathbf{1}''\mathbf{2}''\mathbf{3}'']$ and $[\mathbf{1}''\mathbf{2}''\mathbf{4}'']$, one of which has brackets $[\mathbf{1}''\mathbf{3}''\mathbf{4}'']$ and $[\mathbf{2}''\mathbf{3}''\mathbf{4}'']$, and one of which has brackets $[\mathbf{11}''\mathbf{2}'']$ and $[\mathbf{13}''\mathbf{4}'']$. Similarly, $h_{\mathbf{3}'\mathbf{4}'}$ has nine expansions into bracket polynomials, none of which occurs in $h_{\mathbf{1}'\mathbf{2}'}$.

$h_{\mathbf{1}''\mathbf{2}''}$ has nine expansions into bracket polynomials, two of which occur in $h_{\mathbf{1}'\mathbf{2}'}$ and $h_{\mathbf{3}'\mathbf{4}'}$ respectively, one of which has brackets $[\mathbf{11}'\mathbf{2}']$ and $[\mathbf{13}'\mathbf{4}']$. So there are seven new expansions. Similarly, $h_{\mathbf{3}''\mathbf{4}''}$ brings seven new results.

The distributive expansion separating $\mathbf{1}'\mathbf{2}', \mathbf{3}'\mathbf{4}'$ has nine expansions into bracket polynomials, two of which occur in $h_{\mathbf{1}''\mathbf{2}''}$ and $h_{\mathbf{3}''\mathbf{4}''}$ respectively. So there are seven new expansions. Similarly, the other distributive expansion brings seven new results. All together there are $2 \times (9 + 7 + 7) = 46$ different expansions into bracket polynomials. $\square$

Theorem 3.4. *Let $\{\mathbf{1}, \mathbf{1}', \ldots, \mathbf{4}', \mathbf{1}'', \ldots, \mathbf{4}''\}$ be a set of generic points in the projective plane. Let p_{III} be defined in* (3.8)*.*

1. *Trivial zero. If one of the following conditions is satisfied, p_{III} is trivially zero:* (1) *one of the pairs,* $\mathbf{1'2'}, \mathbf{3'4'}, \mathbf{1''2''}, \mathbf{3''4''}$, *is identical points;* (2) *one of the pairs,* $\{\mathbf{1'2'}, \mathbf{3'4'}\}$ *and* $\{\mathbf{1''2''}, \mathbf{3''4''}\}$, *is identical;* (3) *the two sets* $\{\mathbf{1'2'}, \mathbf{3'4'}\}$ *and* $\{\mathbf{1''2''}, \mathbf{3''4''}\}$ *are identical.*

   In the following, assume that p_{III} is not trivially zero.
2. *Inner intersection. If $\mathbf{1'} = \mathbf{3'}$, point $\mathbf{1'}$ is called an* inner intersection *in p_{III}. The following is a shortest expansion:*
$$[\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{1'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] = [\mathbf{1'2'4'}]\mathbf{11'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}. \tag{3.11}$$
   It is the unique factored expansion. It is the unique shortest expansion if $\mathbf{2'}, \mathbf{4'}$ are not in $\{\mathbf{1''}, \mathbf{2''}\}, \{\mathbf{3''}, \mathbf{4''}\}$ respectively. In the exceptional case, the other shortest expansion is
$$\begin{aligned}[\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{1'4'})(\mathbf{2'2''} \wedge \mathbf{4'4''})] = {} & [\mathbf{11'4'}][\mathbf{1'2'2''}][\mathbf{2'4'4''}] \\ & -[\mathbf{11'2'}][\mathbf{1'4'4''}][\mathbf{2'4'2''}].\end{aligned} \tag{3.12}$$
3. *Double line. If $\mathbf{1'} = \mathbf{1''}, \mathbf{2'} = \mathbf{2''}$, line $\mathbf{1'2'}$ is called a* double line *in p_{III}. The following is the unique shortest expansion:*
$$[\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1'2'} \wedge \mathbf{3''4''})] = [\mathbf{11'2'}]\mathbf{1'2'} \wedge \mathbf{3'4'} \wedge \mathbf{3''4''}. \tag{3.13}$$
   It is also the unique factored expansion.
4. *Recursion of $\mathbf{1}$. If $\mathbf{1'} = \mathbf{1}$, we say $\mathbf{1}$* recurs*. Then the following is a shortest expansion:*
$$[\mathbf{1}(\mathbf{12'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] = [\mathbf{13'4'}]\mathbf{12'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}. \tag{3.14}$$
   It is the unique factored expansion. It is the unique shortest expansion if $\mathbf{3'}, \mathbf{4'}$ are not in $\{\mathbf{1''}, \mathbf{2''}\}, \{\mathbf{3''}, \mathbf{4''}\}$ respectively. In the exceptional case, the other shortest expansion is
$$\begin{aligned}[\mathbf{1}(\mathbf{12'} \wedge \mathbf{3'4'})(\mathbf{3'2''} \wedge \mathbf{4'4''})] = {} & [\mathbf{12'4'}][\mathbf{13'2''}][\mathbf{3'4'4''}] \\ & -[\mathbf{12'3'}][\mathbf{14'4''}][\mathbf{3'4'2''}].\end{aligned} \tag{3.15}$$
5. *Other cases. If p_{III} has neither inner intersection nor double line, and $\mathbf{1}$ does not recur, then p_{III} has no factored expansion. The shortest expansions of p_{III} are two, three or four termed if and only if the set $\mathcal{M} = \{\mathbf{1'}, \mathbf{2'}, \mathbf{3'}, \mathbf{4'}\} \cap \{\mathbf{1''}, \mathbf{2''}, \mathbf{3''}, \mathbf{4''}\}$ has at least two elements, has one element or is empty.*

Theorem 3.5. *Let $\mathbf{1}, \mathbf{1'}, \ldots, \mathbf{4'}, \mathbf{1''}, \ldots, \mathbf{4''}$ be points in the projective plane. Let p_{III} be defined in (3.8). Assume that only the bracket computation rules are applied when expanding p_{III}.*

1. *Trivial zero. If one of the following conditions is satisfied, p_{III} is trivially zero:* (1) *one of the pairs,* $\mathbf{1'2'}, \mathbf{3'4'}, \mathbf{1''2''}, \mathbf{3''4''}$, *is identical points;* (2) *one of the* 4-*tuples,* $\{\mathbf{1'}, \mathbf{2'}, \mathbf{3'}, \mathbf{4'}\}$ *and* $\{\mathbf{1''}, \mathbf{2''}, \mathbf{3''}, \mathbf{4''}\}$, *is four collinear points;* (3) *the four lines,* $\mathbf{1'2'}, \mathbf{3'4'}, \mathbf{1''2''}, \mathbf{3''4''}$, *are concurrent;* (4) $\mathbf{1}$ *equals one of* $\mathbf{1'2'} \cap \mathbf{3'4'}$ *and* $\mathbf{1''2''} \cap \mathbf{3''4''}$.

   In the following items, assume that p_{III} is not trivially zero.

2. *Inner intersection. If* $\mathbf{1}', \mathbf{2}', \mathbf{3}'$ *are collinear, point* $\mathbf{3}'$ *is called an* inner intersection *in* p_{III}. *The following is an expansion:*

$$[\mathbf{1}(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')(\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'')] = [\mathbf{1}'\mathbf{2}'\mathbf{4}']\mathbf{1}\mathbf{3}' \wedge \mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}''. \tag{3.16}$$

If the wedge product in (3.16) *has monomial expansion, then the above expansion is a shortest one; else, it is a shortest expansion under the condition that none of the points* $\mathbf{1}', \mathbf{2}', \mathbf{4}'$ *is on two of the three lines* $\mathbf{1}\mathbf{3}', \mathbf{1}''\mathbf{2}'', \mathbf{3}''\mathbf{4}''$.

3. *Outer intersection. If* $\mathbf{1}'' = \mathbf{1}'\mathbf{2}' \cap \mathbf{3}'\mathbf{4}'$, *it is called an* outer intersection *in* p_{III}. *Then*

$$\begin{aligned}[\mathbf{1}(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')(\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'')] &= [\mathbf{1}\mathbf{1}''\mathbf{2}'']\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}' \wedge \mathbf{3}''\mathbf{4}'' \\ &= -[\mathbf{1}''\mathbf{3}''\mathbf{4}'']\mathbf{1}\mathbf{2}'' \wedge \mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}'.\end{aligned} \tag{3.17}$$

If one of the two expansions can be expanded into a monomial, then it is a shortest expansion; else, (3.17) *contains a shortest expansion if* p_{III} *has no inner intersection.*

4. *Double line. If* $\mathbf{1}', \mathbf{2}', \mathbf{1}'', \mathbf{2}''$ *are collinear, line* $\mathbf{1}'\mathbf{2}'$ *is called a* double line *in* p_{III}. *The following are two expansions, and contain a shortest expansion of* p_{III}:

$$\begin{aligned}[\mathbf{1}(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')(\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'')] &= [\mathbf{1}\mathbf{1}'\mathbf{2}']\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}'\mathbf{4}' \wedge \mathbf{3}''\mathbf{4}'' \\ &= [\mathbf{1}\mathbf{1}''\mathbf{2}'']\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}' \wedge \mathbf{3}''\mathbf{4}''.\end{aligned} \tag{3.18}$$

5. *Recursion of* $\mathbf{1}$. *If* $\mathbf{1}, \mathbf{1}', \mathbf{2}'$ *are collinear, we say point* $\mathbf{1}$ recurs*. Then*

$$[\mathbf{1}(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')(\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'')] = [\mathbf{1}\mathbf{3}'\mathbf{4}']\mathbf{1}'\mathbf{2}' \wedge \mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}''. \tag{3.19}$$

If the wedge product in (3.19) *has monomial expansion, the above expansion is a shortest one; else, it is a shortest expansion under the condition that* p_{III} *has no double line, and none of the points* $\mathbf{1}, \mathbf{3}', \mathbf{4}'$ *is on two of the three lines* $\mathbf{1}'\mathbf{2}', \mathbf{1}''\mathbf{2}'', \mathbf{3}''\mathbf{4}''$.

6. *If* p_{III} *has neither inner intersection, nor outer intersection, nor double line, and point* $\mathbf{1}$ *does not recur, then any expansion of* p_{III} *has at least two terms.*
7. *Under the hypothesis in the previous item, there are only three cases in which* p_{III} *has factored expansions:*

   (a) *Diagonal. If* $\mathbf{1}, \mathbf{1}'', \mathbf{3}''$ *are collinear, we say* $\mathbf{1}$ *is on* diagonal $\mathbf{1}''\mathbf{3}''$. *Then*

$$\begin{aligned}[\mathbf{1}(\mathbf{1}''\mathbf{4}'' \wedge \mathbf{3}'\mathbf{4}')(\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'')] &= [\mathbf{1}''\mathbf{3}''\mathbf{4}''](\mathbf{1}\mathbf{4}' \wedge \mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'' \\ &\quad -[\mathbf{1}\mathbf{2}''\mathbf{3}''][\mathbf{1}''\mathbf{4}''\mathbf{4}']) \\ &= [\mathbf{1}''\mathbf{3}''\mathbf{4}'']([\mathbf{1}\mathbf{1}''\mathbf{4}'][\mathbf{2}''\mathbf{3}''\mathbf{4}''] \\ &\quad -\mathbf{1}\mathbf{2}'' \wedge \mathbf{1}''\mathbf{4}'' \wedge \mathbf{3}''\mathbf{4}').\end{aligned} \tag{3.20}$$

   (b) *Quadrilateral. If* $\mathbf{1} = \mathbf{1}''\mathbf{3}'' \cap \mathbf{2}''\mathbf{4}''$, *then*

$$\begin{aligned}[\mathbf{1}(\mathbf{1}''\mathbf{4}'' \wedge \mathbf{3}'\mathbf{4}')(\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'')] &= [\mathbf{1}\mathbf{1}''\mathbf{4}'']([\mathbf{1}''\mathbf{3}'\mathbf{4}'][\mathbf{2}''\mathbf{3}''\mathbf{4}''] \\ &\quad +[\mathbf{1}''\mathbf{2}''\mathbf{3}''][\mathbf{4}''\mathbf{3}'\mathbf{4}']).\end{aligned} \tag{3.21}$$

   (c) *Complete quadrilateral. If* $\mathbf{1} = \mathbf{1}''\mathbf{3}'' \cap \mathbf{2}''\mathbf{4}''$, *then*

$$[\mathbf{1}(\mathbf{1}''\mathbf{4}'' \wedge \mathbf{2}''\mathbf{3}'')(\mathbf{1}''\mathbf{2}'' \wedge \mathbf{3}''\mathbf{4}'')] = 2[\mathbf{1}\mathbf{1}''\mathbf{4}''][\mathbf{1}''\mathbf{2}''\mathbf{3}''][\mathbf{2}''\mathbf{3}''\mathbf{4}'']. \tag{3.22}$$

3.3. Three intersections of lines

In $[\mathbf{ABC}]$, when $\mathbf{A} = \mathbf{12} \cap \mathbf{34}$, $\mathbf{B} = \mathbf{1'2'} \cap \mathbf{3'4'}$ and $\mathbf{C} = \mathbf{1''2''} \cap \mathbf{3''4''}$, the bracket equals

$$p_{IV} = [(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]. \tag{3.23}$$

Definition 3.5. The *Cayley expansion* of p_{IV} by distributing the pair of lines $\mathbf{12}, \mathbf{34}$, is expanding p_{IV} into a bracket polynomial through

$$\begin{aligned} p_{IV} = {} & (\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{34} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}) \\ & -(\mathbf{34} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{12} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}). \end{aligned} \tag{3.24}$$

There are three distributive expansions of p_{IV}. The *hybrid Cayley expansion* of p_{IV} by separating the pair of points $\mathbf{1}, \mathbf{2}$, is expanding p_{IV} into a bracket polynomial through

$$\begin{aligned} p_{IV} = {} & [\mathbf{134}][\mathbf{2}(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] \\ & -[\mathbf{234}][\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]. \end{aligned} \tag{3.25}$$

There are six hybrid expansions of p_{IV}.

Proposition 3.6. *If all points in p_{IV} are generic ones, then the hybrid expansions and the distributive expansions of p_{IV} produce* 16 847 *different expansions into bracket polynomials.*

Proof. Denote the hybrid expansion separating $\mathbf{i}, \mathbf{j}$ by $h_{\mathbf{ij}}$. By (3.25), $h_{\mathbf{12}}$ has $46^2 = 2116$ different expansions into bracket polynomials. Similarly, $h_{\mathbf{34}}$ has 2116 expansion results, none of which occurs in $h_{\mathbf{12}}$. In $h_{\mathbf{12}}$, there are the following expansions which are also the results of $h_{\mathbf{1'2'}}$:

$$\begin{aligned} & [\mathbf{134}][\mathbf{1'3'4'}]\mathbf{22'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''} - [\mathbf{134}][\mathbf{2'3'4'}]\mathbf{21'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''} \\ & -[\mathbf{234}][\mathbf{1'3'4'}]\mathbf{12'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''} + [\mathbf{234}][\mathbf{2'3'4'}]\mathbf{11'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}. \end{aligned} \tag{3.26}$$

(3.26) has $3^4 = 81$ different expansions into bracket polynomials. It has only one expansion which has $[\mathbf{1''3''4''}]$ and $[\mathbf{2''3''4''}]$.

According to the above account, the hybrid expansions of p_{IV} have altogether $6 \times 2116 - 3 \times 4 \times 81 + 2 \times 4 \times 1 = 11\,732$ different expansions into bracket polynomials.

Now we prove that for generic points $\mathbf{1}, \ldots, \mathbf{6}, \mathbf{1'}, \ldots, \mathbf{6'}$, $p_{II} = (\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56})(\mathbf{1'2'} \wedge \mathbf{3'4'} \wedge \mathbf{5'6'})$ has 45 different expansions into bracket polynomials. From the expansion (3.5), there are nine expansions into bracket polynomials. So p_{II} has $2 \times 3 \times 9 = 54$ expansions, among which $3 \times 3 = 9$ are counted twice. So there are $54 - 9 = 45$ different expansions of p_{II} into bracket polynomials. As a corollary, the number of different expansions into bracket polynomials from (3.24) is $45^2 = 2025$.

The following expansion of (3.24) also belongs to $h_{\mathbf{3'4'}}$:

$$\begin{aligned} & [\mathbf{123'}][\mathbf{1'2'4'}]\mathbf{34} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''} - [\mathbf{124'}][\mathbf{1'2'3'}]\mathbf{34} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''} \\ & -[\mathbf{343'}][\mathbf{1'2'4'}]\mathbf{12} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''} + [\mathbf{344'}][\mathbf{1'2'3'}]\mathbf{12} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}. \end{aligned} \tag{3.27}$$

(3.27) contains $3^4 = 81$ different expansions into bracket polynomials. It has only one expansion which has $[\mathbf{1''3''4''}]$ and $[\mathbf{2''3''4''}]$.

There are three distributive expansions of p_{IV}. No expansion into bracket polynomials can belong to two of the three distributive expansions.

According to the above arguments, there are $3 \times 4 \times 81 = 972$ expansions into bracket polynomials which belong to a distributive expansion and a hybrid expansion of p_{IV}, and there are $3 \times 4 = 12$ different expansions which belong to a distributive expansion and two hybrid expansions of p_{IV}. The three distributive expansions contribute $3 \times 2025 - 972 + 12 = 5115$ new expansions. The total sum of expansions is $11\,732 + 5115 = 16\,847$. $\square$

Theorem 3.7. *Let* $\{\mathbf{1}, \ldots, \mathbf{4}, \mathbf{1}', \ldots, \mathbf{4}', \mathbf{1}'', \ldots, \mathbf{4}''\}$ *be a set of generic points in the projective plane. Let* p_{IV} *be defined in* (3.23).

1. *Trivial zero. If one of the following conditions is satisfied,* p_{IV} *is trivially zero:* (1) *one of the six pairs,* $\mathbf{12}, \mathbf{34}, \mathbf{1'2'}, \mathbf{3'4'}, \mathbf{1''2''}, \mathbf{3''4''}$, *is identical points;* (2) *one of the three pairs,* $\{\mathbf{12}, \mathbf{34}\}, \{\mathbf{1'2'}, \mathbf{3'4'}\}, \{\mathbf{1''2''}, \mathbf{3''4''}\}$, *is identical;* (3) *two of the three sets* $\{\mathbf{12}, \mathbf{34}\}, \{\mathbf{1'2'}, \mathbf{3'4'}\}, \{\mathbf{1''2''}, \mathbf{3''4''}\}$ *are identical.*

   In the following, assume that p_{IV} *is not trivially zero.*

2. *Inner intersection. If* $\mathbf{1} = \mathbf{3}$, *point* $\mathbf{1}$ *is called an inner intersection in* p_{IV}. *The following is the unique factored expansion:*

$$\begin{aligned}&[(\mathbf{12} \wedge \mathbf{14})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]\\ &\quad = [\mathbf{124}][\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]. \qquad (3.28)\end{aligned}$$

   It is also a shortest expansion.

3. *Double line. If* $\mathbf{12} = \mathbf{1'2'}$, *line* $\mathbf{12}$ *is called a* double line *in* p_{IV}. *The following is the unique factored expansion:*

$$\begin{aligned}&[(\mathbf{12} \wedge \mathbf{34})(\mathbf{12} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]\\ &\quad = (\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{3'4'})(\mathbf{12} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}). \qquad (3.29)\end{aligned}$$

   (3.29) *is a shortest expansion if either it has an expansion of two terms, or* p_{IV} *has another double line, or the set* $\mathcal{N} = \{\mathbf{3}, \mathbf{4}, \mathbf{3}', \mathbf{4}'\} \cap \{\mathbf{1}'', \mathbf{2}'', \mathbf{3}'', \mathbf{4}''\}$ *has at most two elements. When neither is satisfied, the shortest expansions of* p_{IV} *have three terms if and only if* $\mathcal{N}$ *has three elements.*

4. *Triangle. If* $\mathbf{1}' = \mathbf{1}$ *and* $\mathbf{1''2''} = \mathbf{22'}$, *then* $\mathbf{122'}$ *is called a triangle of* p_{IV}. *We have*

$$\begin{aligned}[(\mathbf{12} \wedge \mathbf{34})\,(\mathbf{12'} \wedge \mathbf{3'4'})\,(\mathbf{22'} \wedge \mathbf{3''4''})] = [\mathbf{122'}]([\mathbf{134}][\mathbf{23''4''}][\mathbf{2'3'4'}]\\ -[\mathbf{13'4'}][\mathbf{234}][\mathbf{2'3''4''}]). \qquad (3.30)\end{aligned}$$

   When there is neither inner intersection nor double line, the triangle pattern has three further factorable subpatterns:

   (a) *Complete quadrilateral. If* $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\} = \{\mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{4}'\} = \{\mathbf{1}'', \mathbf{2}'', \mathbf{3}'', \mathbf{4}''\}$, *then*

$$[(\mathbf{12} \wedge \mathbf{34})(\mathbf{13} \wedge \mathbf{24})(\mathbf{14} \wedge \mathbf{23})] = -2[\mathbf{123}][\mathbf{124}][\mathbf{134}][\mathbf{234}]. \qquad (3.31)$$

   The monomial expression is symmetric with respect to $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$.

(b) *Quadrilateral. If p_{IV} has two triangles sharing the same side, for example triangles* **123** *and* **134***, then the only possible pattern is*

$$[(\mathbf{12} \wedge \mathbf{34})\ (\mathbf{13} \wedge \mathbf{24})\ (\mathbf{14} \wedge \mathbf{3''4''})] = -[\mathbf{124}][\mathbf{134}]([\mathbf{123}][\mathbf{43''4''}] + [\mathbf{13''4''}][\mathbf{234}]). \tag{3.32}$$

$(\mathbf{1234}, \mathbf{14})$ *is called a* quadrilateral *of p_{IV}.*

(c) *Triangle pair. If p_{IV} has two disentangled triangles, the only possible pattern is*

$$[(\mathbf{12} \wedge \mathbf{34})\ (\mathbf{12'} \wedge \mathbf{34'})\ (\mathbf{22'} \wedge \mathbf{44'})] = -[\mathbf{122'}][\mathbf{344'}]\mathbf{13} \wedge \mathbf{24} \wedge \mathbf{2'4'}. \tag{3.33}$$

$(\mathbf{122'}, \mathbf{344'})$ *is called a* triangle pair *of p_{IV}.*

5. *If p_{IV} is not trivially zero, has neither inner intersection, nor double line, nor triangle, then it has no factored expansion. In the following items, the above hypothesis is always assumed. A* single, double *or* triple *point of p_{IV} is a point that occurs once, twice or three times in it.*
6. *Two triple points. If p_{IV} has two triple points, then it has* 2*-termed expansion.*
7. *One triple point and two double points. If p_{IV} has only one triple point* **1***, it has* 2*-termed expansion if and only if there are two double points* $\mathbf{2}, \mathbf{3}$ *such that p_{IV} is of the form* $[(\mathbf{14} \wedge \mathbf{23})(\mathbf{12'} \wedge \mathbf{3'4'})(\mathbf{12''} \wedge \mathbf{3''4''})]$*, where* $\mathbf{2} \in \{\mathbf{2'}, \mathbf{3'}, \mathbf{4'}\}$ *and* $\mathbf{3} \in \{\mathbf{2''}, \mathbf{3''}, \mathbf{4''}\}$.
8. *Four double points. If p_{IV} has no triple point, it has* 2*-termed expansion if and only if there are four double points* $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ *such that p_{IV} is in one of the following forms:*

(a) 4-2-2 *pattern.* $[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]$*, where* $\{\mathbf{1}, \mathbf{3}\} \subseteq \{\mathbf{1'}, \ldots, \mathbf{4'}\}$ *and* $\{\mathbf{2}, \mathbf{4}\} \subseteq \{\mathbf{1''}, \ldots, \mathbf{4''}\}$.

(b) 3-3-2 *pattern.* $[(\mathbf{12} \wedge \mathbf{56})(\mathbf{13} \wedge \mathbf{46'})(\mathbf{24} \wedge \mathbf{36''})]$*, where* $\mathbf{5}, \mathbf{6}, \mathbf{6'}, \mathbf{6''}$ *are either single or double points.*

Remark. (3.32) comes from the following identity:

$$\begin{aligned}(\mathbf{12} \wedge \mathbf{34})(\mathbf{13} \wedge \mathbf{24}) &= -[\mathbf{134}][\mathbf{234}]\mathbf{12} + [\mathbf{123}][\mathbf{124}]\mathbf{34} \\ &= -[\mathbf{124}][\mathbf{234}]\mathbf{13} + [\mathbf{123}][\mathbf{134}]\mathbf{24}.\end{aligned} \tag{3.34}$$

The expression is antisymmetric with respect to $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$.

When there are collinear constraints among the points in p_{IV}, the complete classification of factored expansions and 2-termed expansions of p_{IV} is extremely difficult. Below we provide some typical patterns with factored expansions.

Proposition 3.8. *Let* $\mathbf{1}, \ldots, \mathbf{4}, \mathbf{1'}, \ldots, \mathbf{4'}, \mathbf{1''}, \ldots, \mathbf{4''}$ *be points in the projective plane, and assume that only the bracket computation rules are applied when expanding p_{IV} in* (3.23).

1. *Inner intersection. If* $\mathbf{1}, \mathbf{2}, \mathbf{3}$ *are collinear, then*

$$\begin{aligned}&[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] \\ &\quad = [\mathbf{124}][\mathbf{3}(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})].\end{aligned} \tag{3.35}$$

2. *Outer intersection. If* $\mathbf{1}' = \mathbf{12} \cap \mathbf{34}$, *then*

$$\begin{aligned}
&[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] \\
&\quad = (\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{3'4'})(\mathbf{1'2'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}) \\
&\quad = -[\mathbf{1'3'4'}][\mathbf{2'}(\mathbf{12} \wedge \mathbf{34})(\mathbf{1''2''} \wedge \mathbf{3''4''})].
\end{aligned} \tag{3.36}$$

3. *Double line. If* $\mathbf{1}, \mathbf{2}, \mathbf{1}', \mathbf{2}'$ *are collinear, then*

$$\begin{aligned}
&[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] \\
&\quad = (\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{3'4'})(\mathbf{1'2'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}) \\
&\quad = (\mathbf{1'2'} \wedge \mathbf{34} \wedge \mathbf{3'4'})(\mathbf{12} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}).
\end{aligned} \tag{3.37}$$

4. *Generalized triangle. If* $\mathbf{1}, \mathbf{1}', \mathbf{2}', \mathbf{1}''$ *are collinear, and* $\mathbf{2}, \mathbf{1}'', \mathbf{2}''$ *are collinear, then*

$$\begin{aligned}
[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] = {}& [\mathbf{1'2'2''}]([\mathbf{134}][\mathbf{23''4''}][\mathbf{3'4'1''}] \\
& - [\mathbf{13'4'}][\mathbf{234}][\mathbf{1''3''4''}]).
\end{aligned} \tag{3.38}$$

The following are some further factorable generalized triangle patterns:

(a) *Generalized complete quadrilateral. If* $\mathbf{1}, \mathbf{3}, \mathbf{1}', \mathbf{2}'$ *are collinear, and* $\mathbf{2}, \mathbf{3}, \mathbf{2}''$ *are collinear, then*

$$[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{24})(\mathbf{32''} \wedge \mathbf{14})] = -2[\mathbf{1'2'2''}][\mathbf{124}][\mathbf{134}][\mathbf{234}]. \tag{3.39}$$

(b) *Generalized quadrilateral. If* $\mathbf{1}, \mathbf{1}', \mathbf{2}', \mathbf{1}''$ *are collinear, and* $\mathbf{2}, \mathbf{1}'', \mathbf{2}''$ *are collinear, then*

$$\begin{aligned}
[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{24'})(\mathbf{1''2''} \wedge \mathbf{14'})] = {}& [\mathbf{1'2'2''}][\mathbf{124'}]([\mathbf{134}][\mathbf{24'1''}] \\
& + [\mathbf{14'1''}][\mathbf{234}]).
\end{aligned} \tag{3.40}$$

(c) *Generalized triangle pair. If* $\mathbf{1}, \mathbf{1}', \mathbf{2}', \mathbf{1}''$ *are collinear, and* $\mathbf{2}, \mathbf{1}'', \mathbf{2}''$ *are collinear, then*

$$\begin{aligned}
&[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{34'})(\mathbf{1''2''} \wedge \mathbf{44'})] \\
&\quad = [\mathbf{1'2'2''}][\mathbf{344'}]\mathbf{13} \wedge \mathbf{24} \wedge \mathbf{4'1''}.
\end{aligned} \tag{3.41}$$

(d) *Perspective triangle. If* $\mathbf{1}, \mathbf{3}, \mathbf{2}''$ *are collinear, then*

$$\begin{aligned}
[(\mathbf{12} \wedge \mathbf{34})(\mathbf{13} \wedge \mathbf{24'})(\mathbf{22''} \wedge \mathbf{14'})] = {}& [\mathbf{123}][\mathbf{124'}]([\mathbf{12''4'}][\mathbf{234}] \\
& + [\mathbf{134}][\mathbf{22''4'}]).
\end{aligned} \tag{3.42}$$

5. *Perspective pattern. If* $\mathbf{1}, \mathbf{3}, \mathbf{2}''$ *are collinear, then*

$$\begin{aligned}
[(\mathbf{12} \wedge \mathbf{34})(\mathbf{13} \wedge \mathbf{3'4'})(\mathbf{22''} \wedge \mathbf{3''4''})] = {}& [\mathbf{123}]([\mathbf{134}][\mathbf{23''4''}][\mathbf{2''3'4'}] \\
& - [\mathbf{13'4'}][\mathbf{234}][\mathbf{2''3''4''}]).
\end{aligned} \tag{3.43}$$

6. *Double perspective pattern. If* $\mathbf{2}, \mathbf{3}, \mathbf{1}''$ *are collinear, and* $\mathbf{1}, \mathbf{2}, \mathbf{2}''$ *are collinear, then*

$$\begin{aligned}
&[(\mathbf{12} \wedge \mathbf{34})(\mathbf{13} \wedge \mathbf{24'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] \\
&\quad = [\mathbf{123}]([\mathbf{124'}][\mathbf{1''3''4''}][\mathbf{2''34}] - [\mathbf{124}][\mathbf{1''34'}][\mathbf{2''3''4''}] \\
&\qquad + [\mathbf{2''3''4''}]\mathbf{13} \wedge \mathbf{24'} \wedge \mathbf{1''4}).
\end{aligned} \tag{3.44}$$

Remark. The equality (3.41) cannot be realized by ANY expansion. It can be obtained by the degree-3 Cayley factorization algorithm in Section 4.

3.4. Semifree points and their conjugates

Let $\mathbf{C}$ be a point on line $\mathbf{AB}$. Then $[\mathbf{AB}]\mathbf{C} = [\mathbf{AC}]\mathbf{B} - [\mathbf{BC}]\mathbf{A}$. The *harmonic conjugate*, or simply *conjugate*, of $\mathbf{C}$ with respect to $\mathbf{A}$, $\mathbf{B}$, is a point $\mathbf{D}$ on line $\mathbf{AB}$ such that the cross ratio

$$(\mathbf{AB}; \mathbf{CD}) = \frac{[\mathbf{AC}][\mathbf{BD}]}{[\mathbf{AD}][\mathbf{BC}]} = -1. \tag{3.45}$$

The conjugate of $\mathbf{C}$ with respect to $\mathbf{A}$, $\mathbf{B}$ is

$$\mathrm{conjugate}_{\mathbf{AB}}(\mathbf{C}) = [\mathbf{AC}]\mathbf{B} + [\mathbf{BC}]\mathbf{A}. \tag{3.46}$$

We call a free point on a line a *semifree point* in the plane. Let $\mathbf{A_{56}}, \mathbf{A_{5'6'}}, \mathbf{A_{5''6''}}$ be semifree points (or conjugates of semifree points) on lines $\mathbf{56}, \mathbf{5'6'}, \mathbf{5''6''}$ respectively. There are six kinds of new brackets formed by free points, semifree points and intersections:

$$\begin{array}{lll} [\mathbf{11'A_{5''6''}}], & [\mathbf{1A_{5'6'}A_{5''6''}}], & [\mathbf{AA_{5'6'}A_{5''6''}}], \\ [\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{5''6''}}], & [(\mathbf{12} \wedge \mathbf{34})\mathbf{A_{5'6'}A_{5''6''}}], & [(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{5''6''}}]. \end{array}$$

Since the expansions of the former three brackets are unique, only the latter three brackets need further investigation.

Proposition 3.9. *Let* $\{\mathbf{1}, \mathbf{1'}, \mathbf{2'}, \mathbf{3'}, \mathbf{4'}, \mathbf{5''}, \mathbf{6''}\}$ *be a set of generic points. Let* $\mathbf{A_{5''6''}} = \lambda_{\mathbf{6''}}\mathbf{5''} + \lambda_{\mathbf{5''}}\mathbf{6''}$, *where* $\lambda_{\mathbf{5''}}, \lambda_{\mathbf{6''}}$ *are generic polynomials, and* $\mathbf{5''} \neq \mathbf{6''}$. *Let*

$$q_I = [\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{5''6''}}]. \tag{3.47}$$

The Cayley expansions *of* q_I *are those of*

$$-\lambda_{\mathbf{6''}}\mathbf{15''} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'} - \lambda_{\mathbf{5''}}\mathbf{16''} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'}. \tag{3.48}$$

1. *Trivial zero.* q_I *is trivially zero if one of the following conditions is satisfied:* (1) *one of the pairs,* $\mathbf{1'2'}, \mathbf{3'4'}$, *is identical points;* (2) $\{\mathbf{1'}, \mathbf{2'}\} = \{\mathbf{3'}, \mathbf{4'}\}$.
   Below we assume that q_I *is not trivially zero.*
2. *Inner intersection. If* $\mathbf{1'} = \mathbf{3'}$, *the following is the unique shortest expansion:*
$$[\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{1'4'})\mathbf{A_{5''6''}}] = [\mathbf{1'2'4'}][\mathbf{11'A_{5''6''}}]. \tag{3.49}$$
3. *Double line. If* $\{\mathbf{1'}, \mathbf{2'}\} = \{\mathbf{5''}, \mathbf{6''}\}$, *the following is the unique shortest expansion:*
$$[\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{1'2'}}] = [\mathbf{11'2'}][\mathbf{3'4'A_{1'2'}}]. \tag{3.50}$$
4. *Recursion of* $\mathbf{1}$. *If* $\mathbf{1} = \mathbf{1'}$, *the following is the unique shortest expansion:*
$$[\mathbf{1}(\mathbf{12'} \wedge \mathbf{3'4'})\mathbf{A_{5''6''}}] = [\mathbf{13'4'}][\mathbf{12'A_{5''6''}}]. \tag{3.51}$$
   If $\mathbf{1} = \mathbf{5''}$, *the following is the unique shortest expansion:*
$$[\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{16''}}] = -\lambda_{\mathbf{1}}\mathbf{16''} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'}. \tag{3.52}$$

5. *Other cases. If q_I is not trivially zero, has neither inner intersection nor double line, and* $\mathbf{1}$ *does not recur, then it has no factored expansion. Its shortest expansions have* 2, 3 *or* 4 *terms if and only if* $\{\mathbf{5}'', \mathbf{6}''\} \cap \{\mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{4}'\}$ *has* 2, 1 *or* 0 *elements.*

Proposition 3.10. *Let* $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}', \mathbf{6}', \mathbf{5}'', \mathbf{6}''\}$ *be a set of generic points. Let* $\mathbf{A}_{\mathbf{5}'\mathbf{6}'} = \lambda_{\mathbf{6}'}\mathbf{5}' + \lambda_{\mathbf{5}'}\mathbf{6}'$, $\mathbf{B}_{\mathbf{5}''\mathbf{6}''} = \mu_{\mathbf{6}''}\mathbf{5}'' + \mu_{\mathbf{5}''}\mathbf{6}''$, *where the* λ*'s and* μ*'s are generic polynomials,* $\mathbf{5}' \neq \mathbf{6}'$ *and* $\mathbf{5}'' \neq \mathbf{6}''$. *Let*

$$q_{II} = [(\mathbf{12} \wedge \mathbf{34})\mathbf{A}_{\mathbf{5}'\mathbf{6}'}\mathbf{B}_{\mathbf{5}''\mathbf{6}''}]. \tag{3.53}$$

The Cayley expansions *of* q_{II} *are those of*

$$\lambda_{\mathbf{6}'}\mu_{\mathbf{6}''}\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{5}'\mathbf{5}'' + \lambda_{\mathbf{6}'}\mu_{\mathbf{5}''}\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{5}'\mathbf{6}'' \\ +\lambda_{\mathbf{5}'}\mu_{\mathbf{6}''}\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{6}'\mathbf{5}'' + \lambda_{\mathbf{5}'}\mu_{\mathbf{5}''}\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{6}'\mathbf{6}''. \tag{3.54}$$

1. *Trivial zero.* q_{II} *is trivially zero if one of the following conditions is satisfied:* (1) *one of the pairs,* $\mathbf{12}, \mathbf{34}$, *is identical points;* (2) $\{\mathbf{1}, \mathbf{2}\} = \{\mathbf{3}, \mathbf{4}\}$.
   Below we assume that q_{II} *is not trivially zero.*
2. *Inner intersection. If* $\mathbf{1} = \mathbf{3}$, *the following is a shortest expansion:*

$$[(\mathbf{12} \wedge \mathbf{14})\mathbf{A}_{\mathbf{5}'\mathbf{6}'}\mathbf{B}_{\mathbf{5}''\mathbf{6}''}] = [\mathbf{124}][\mathbf{1}\mathbf{A}_{\mathbf{5}'\mathbf{6}'}\mathbf{B}_{\mathbf{5}''\mathbf{6}''}]. \tag{3.55}$$

3. *Double line. If* $\{\mathbf{5}', \mathbf{6}'\} = \{\mathbf{5}'', \mathbf{6}''\}$, *the following is the unique expansion:*

$$[(\mathbf{12} \wedge \mathbf{34})\mathbf{A}_{\mathbf{5}'\mathbf{6}'}\mathbf{B}_{\mathbf{5}'\mathbf{6}'}] = (\lambda_{\mathbf{6}'}\mu_{\mathbf{5}'} - \lambda_{\mathbf{5}'}\mu_{\mathbf{6}'})\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{5}'\mathbf{6}'. \tag{3.56}$$

   If $\{\mathbf{1}, \mathbf{2}\} = \{\mathbf{5}', \mathbf{6}'\}$, *the following is a shortest expansion:*

$$[(\mathbf{12} \wedge \mathbf{34})\mathbf{A}_{\mathbf{12}}\mathbf{B}_{\mathbf{5}''\mathbf{6}''}] = [\mathbf{12}\mathbf{B}_{\mathbf{5}''\mathbf{6}''}][\mathbf{34}\mathbf{A}_{\mathbf{12}}]. \tag{3.57}$$

4. *Triangle. If* $\mathbf{5}' = \mathbf{1}$ *and* $\{\mathbf{5}'', \mathbf{6}''\} = \{\mathbf{2}, \mathbf{6}'\}$, *then* $\mathbf{126}'$ *is called a triangle in* q_{II}. *The following is a shortest expansion:*

$$[(\mathbf{12} \wedge \mathbf{34})\mathbf{A}_{\mathbf{16}'}\mathbf{B}_{\mathbf{26}'}] = [\mathbf{126}'](\lambda_{\mathbf{1}}\mu_{\mathbf{6}'}[\mathbf{234}] - \lambda_{\mathbf{6}'}\mu_{\mathbf{2}}[\mathbf{134}]). \tag{3.58}$$

5. *Other cases. If* q_{II} *is not trivially zero, has neither inner intersection, nor double line, nor triangle, then it has no factored expansion.* q_{II} *has* 2*-termed expansion if and only if it is of the form* $[(\mathbf{12} \wedge \mathbf{34})\mathbf{A}_{\mathbf{13}}\mathbf{B}_{\mathbf{24}}]$. *The unique* 2*-termed expansion is*

$$[(\mathbf{12} \wedge \mathbf{34})\mathbf{A}_{\mathbf{13}}\mathbf{B}_{\mathbf{24}}] = [\mathbf{12}\mathbf{A}_{\mathbf{13}}][\mathbf{34}\mathbf{B}_{\mathbf{24}}] - [\mathbf{12}\mathbf{B}_{\mathbf{24}}][\mathbf{34}\mathbf{A}_{\mathbf{13}}]. \tag{3.59}$$

Proposition 3.11. *Let* $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{4}', \mathbf{5}'', \mathbf{6}''\}$ *be a set of generic points. Let* $\mathbf{A}_{\mathbf{5}''\mathbf{6}''} = \lambda_{\mathbf{6}''}\mathbf{5}'' + \lambda_{\mathbf{5}''}\mathbf{6}''$, *where* $\lambda_{\mathbf{5}''}, \lambda_{\mathbf{6}''}$ *are generic polynomials, and* $\mathbf{5}'' \neq \mathbf{6}''$. *Let*

$$q_{III} = [(\mathbf{12} \wedge \mathbf{34})(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')\mathbf{A}_{\mathbf{5}''\mathbf{6}''}]. \tag{3.60}$$

The Cayley expansions *of* q_{III} *are those of*

$$\lambda_{\mathbf{6}''}[\mathbf{5}''(\mathbf{12} \wedge \mathbf{34})(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')] + \lambda_{\mathbf{5}''}[\mathbf{6}''(\mathbf{12} \wedge \mathbf{34})(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')]. \tag{3.61}$$

1. *Trivial zero.* q_{III} *is trivially zero if one of the following conditions is satisfied:* (1) *one of the pairs,* $\mathbf{12}, \mathbf{34}$, *is identical points;* (2) $\{\mathbf{1}, \mathbf{2}\} = \{\mathbf{3}, \mathbf{4}\}$ *or* $\{\mathbf{1}', \mathbf{2}'\} = \{\mathbf{3}', \mathbf{4}'\}$; (3) $\{\mathbf{12}, \mathbf{34}\} = \{\mathbf{1}'\mathbf{2}', \mathbf{3}'\mathbf{4}'\}$.
   Below we assume that q_{III} *is not trivially zero.*

2. *Inner intersection. If* $\mathbf{1} = \mathbf{3}$*, the following is a shortest expansion:*

$$[(\mathbf{12} \wedge \mathbf{14})(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{5''6''}}] = -[\mathbf{124}]\mathbf{1}\mathbf{A_{5''6''}} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'}. \tag{3.62}$$

3. *Double line. If* $\{\mathbf{1}, \mathbf{2}\} = \{\mathbf{1'}, \mathbf{2'}\}$*, the following is a shortest expansion:*

$$[(\mathbf{12} \wedge \mathbf{34})(\mathbf{12} \wedge \mathbf{3'4'})\mathbf{A_{5''6''}}] = [\mathbf{12}\mathbf{A_{5''6''}}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{3'4'}. \tag{3.63}$$

If $\{\mathbf{1}, \mathbf{2}\} = \{\mathbf{5''}, \mathbf{6''}\}$*, the following is a shortest expansion:*

$$[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{12}}] = [\mathbf{34}\mathbf{A_{12}}]\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'}. \tag{3.64}$$

4. *Triangle. If* $\mathbf{1'} = \mathbf{1}$ *and* $\{\mathbf{5''}, \mathbf{6''}\} = \{\mathbf{2}, \mathbf{2'}\}$*, then* $\mathbf{122'}$ *is called a* triangle *in* q_{III}*. The following is a shortest expansion:*

$$\begin{aligned}[(\mathbf{12} \wedge \mathbf{34})(\mathbf{12'} \wedge \mathbf{3'4'})\mathbf{A_{22'}}] = [\mathbf{122'}](&\lambda_{\mathbf{2'}}[\mathbf{13'4'}][\mathbf{234}] \\ &+ \lambda_{\mathbf{2}}[\mathbf{134}][\mathbf{2'3'4'}]).\end{aligned} \tag{3.65}$$

The triangle pattern has one further factorable subpattern:

Quadrilateral. $(\mathbf{1234}, \mathbf{14})$ *is called a* quadrilateral *in* $[(\mathbf{12} \wedge \mathbf{34})(\mathbf{13} \wedge \mathbf{24})\mathbf{A_{14}}]$*. The following is a shortest expansion:*

$$[(\mathbf{12} \wedge \mathbf{34})(\mathbf{13} \wedge \mathbf{24})\mathbf{A_{14}}] = [\mathbf{124}][\mathbf{134}](\lambda_{\mathbf{4}}[\mathbf{123}] - \lambda_{\mathbf{1}}[\mathbf{234}]). \tag{3.66}$$

5. *If* q_{III} *is not trivially zero, has neither inner intersection, nor double line, nor triangle, then it has no factored expansion.*

   In the following, the above hypothesis is always assumed. A triple, double *or* single *point of* q_{III} *is a point that occurs three times, twice or once in the sequence* $\mathbf{1}, \ldots, \mathbf{4}, \mathbf{1'}, \ldots, \mathbf{4'}, \mathbf{5''}, \mathbf{6''}$.

6. *(Two triple points). If* q_{III} *has two triple points, then it has* 2*-termed expansion.*
7. *(One triple point and two double points). If* q_{III} *has only one triple point* $\mathbf{1}$*, then it has* 2*-termed expansion if and only if there are two double points* $\mathbf{2}, \mathbf{3}$ *such that* q_{III} *is of the form* $[(\mathbf{14} \wedge \mathbf{23})(\mathbf{15} \wedge \mathbf{36})\mathbf{A_{12}}]$*, where* $\mathbf{4}, \mathbf{5}, \mathbf{6}$ *are double or single points.*
8. *(Four double points). If* q_{III} *has no triple point, then it has* 2*-termed expansion if and only if there are four double points* $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ *such that* q_{III} *is in one of the following forms, where* $\mathbf{5}, \mathbf{6}$ *are either single or double points:*

   (a) 4-2-2 *pattern.* $[(\mathbf{12} \wedge \mathbf{34})(\mathbf{24} \wedge \mathbf{56})\mathbf{A_{13}}]$.
   (b) 3-3-2 *pattern.* $[(\mathbf{12} \wedge \mathbf{45})(\mathbf{24} \wedge \mathbf{36})\mathbf{A_{13}}]$.

3.5. Cayley expansion in three dimensions

For line $\mathbf{12}$ and planes $\mathbf{1'2'3'}$, $\mathbf{1''2''3''}$, there is the following wedge product:

$$r_I = \mathbf{12} \wedge \mathbf{1'2'3'} \wedge \mathbf{1''2''3''}. \tag{3.67}$$

r_I has three different *Cayley expansions*. The expansion which distributes $\mathbf{1}, \mathbf{2}$ is

$$\mathbf{12} \wedge \mathbf{1'2'3'} \wedge \mathbf{1''2''3''} = [\mathbf{11'2'3'}][\mathbf{21''2''3''}] - [\mathbf{21'2'3'}][\mathbf{11''2''3''}]. \tag{3.68}$$

The expansion which distributes $\mathbf{1}', \mathbf{2}', \mathbf{3}'$ is

$$\begin{aligned}\mathbf{12} \wedge \mathbf{1}'\mathbf{2}'\mathbf{3}' \wedge \mathbf{1}''\mathbf{2}''\mathbf{3}'' = [\mathbf{121}'\mathbf{2}'][\mathbf{3}'\mathbf{1}''\mathbf{2}''\mathbf{3}''] - [\mathbf{121}'\mathbf{3}'][\mathbf{2}'\mathbf{1}''\mathbf{2}''\mathbf{3}''] \\ + [\mathbf{122}'\mathbf{3}'][\mathbf{1}'\mathbf{1}''\mathbf{2}''\mathbf{3}''].\end{aligned} \tag{3.69}$$

Proposition 3.12. *Let* $\{\mathbf{1}, \mathbf{2}, \mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{1}'', \mathbf{2}'', \mathbf{3}''\}$ *be generic points in the projective space. Let* r_I *be defined by* (3.67).

1. *Zero.* $r_I = 0$ *by expansion if and only if one of the following conditions is satisfied:* (1) *one of the tuples,* $\mathbf{12}, \mathbf{1}'\mathbf{2}'\mathbf{3}', \mathbf{1}''\mathbf{2}''\mathbf{3}''$*, contains two identical points;* (2) *the two* 3*-tuples are identical;* (3) *the* 2*-tuple is in one of the* 3*-tuples;* (4) *the three tuples have a point in common.*

   In the following, assume that $r_I \neq 0$ *by expansion.*
2. *Recursion of* $\mathbf{1}$*. If* $\mathbf{1} = \mathbf{1}'$*, then* $\mathbf{12} \wedge \mathbf{12}'\mathbf{3}' \wedge \mathbf{1}''\mathbf{2}''\mathbf{3}'' = [\mathbf{122}'\mathbf{3}'][\mathbf{11}''\mathbf{2}''\mathbf{3}'']$.
3. *Double line. If* $\{\mathbf{1}', \mathbf{2}'\} = \{\mathbf{1}'', \mathbf{2}''\}$*, then* $\mathbf{12} \wedge \mathbf{1}'\mathbf{2}'\mathbf{3}' \wedge \mathbf{1}'\mathbf{2}'\mathbf{3}'' = [\mathbf{1}'\mathbf{2}'\mathbf{3}'\mathbf{3}''][\mathbf{121}'\mathbf{2}']$.
4. *If* r_I *has neither recursive point nor double line, then it has no factored expansion.*

For planes $\mathbf{123}, \mathbf{1}'\mathbf{2}'\mathbf{3}', \mathbf{1}''\mathbf{2}''\mathbf{3}'', \mathbf{1}'''\mathbf{2}'''\mathbf{3}'''$*, there is the following wedge product:*

$$r_{II} = \mathbf{123} \wedge \mathbf{1}'\mathbf{2}'\mathbf{3}' \wedge \mathbf{1}''\mathbf{2}''\mathbf{3}'' \wedge \mathbf{1}'''\mathbf{2}'''\mathbf{3}'''. \tag{3.70}$$

r_{II} *has* 12 *different* Cayley expansions*. The expansion which distributes* $\mathbf{1}, \mathbf{2}, \mathbf{3}$ *towards* $\mathbf{1}'\mathbf{2}'\mathbf{3}'$ *is*

$$\begin{aligned}r_{II} = [\mathbf{11}'\mathbf{2}'\mathbf{3}']\mathbf{23} \wedge \mathbf{1}''\mathbf{2}''\mathbf{3}'' \wedge \mathbf{1}'''\mathbf{2}'''\mathbf{3}''' - [\mathbf{21}'\mathbf{2}'\mathbf{3}']\mathbf{13} \wedge \mathbf{1}''\mathbf{2}''\mathbf{3}'' \wedge \mathbf{1}'''\mathbf{2}'''\mathbf{3}''' \\ + [\mathbf{31}'\mathbf{2}'\mathbf{3}']\mathbf{12} \wedge \mathbf{1}''\mathbf{2}''\mathbf{3}'' \wedge \mathbf{1}'''\mathbf{2}'''\mathbf{3}'''.\end{aligned} \tag{3.71}$$

Proposition 3.13. *Let* $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{1}''', \mathbf{2}''', \mathbf{3}'''\}$ *be a set of generic points. Let* r_{II} *be defined in* (3.70).

1. *Trivial zero. If one of the following conditions is satisfied,* r_{II} *is trivially zero:* (1) *one of the* 3*-tuples,* $\mathbf{123}, \mathbf{1}'\mathbf{2}'\mathbf{3}', \mathbf{1}''\mathbf{2}''\mathbf{3}'', \mathbf{1}'''\mathbf{2}'''\mathbf{3}'''$*, contains two identical points;* (2) *two of the four* 3*-tuples are identical;* (3) *three of the four* 3*-tuples have two points in common;* (4) *the four* 3*-tuples have a point in common.*

   In the following, assume that r_{II} *is not trivially zero.*
2. *Triple point. If* $\mathbf{1} = \mathbf{1}' = \mathbf{1}''$*, then* $\mathbf{1}$ *is called a* triple point *in* r_{II}*. Denote by* $\mathbf{23} \wedge_{\mathbf{1}} \mathbf{2}'\mathbf{3}' \wedge_{\mathbf{1}} \mathbf{2}''\mathbf{3}''$ *the results of changing the bracket expansions of* $\mathbf{23} \wedge \mathbf{2}'\mathbf{3}' \wedge \mathbf{2}''\mathbf{3}''$ *from* $[\mathbf{ijk}]$ *to* $[\mathbf{1ijk}]$*. Then*

$$\mathbf{123} \wedge \mathbf{12}'\mathbf{3}' \wedge \mathbf{12}''\mathbf{3}'' \wedge \mathbf{1}'''\mathbf{2}'''\mathbf{3}''' = [\mathbf{11}'''\mathbf{2}'''\mathbf{3}''']\, \mathbf{23} \wedge_{\mathbf{1}} \mathbf{2}'\mathbf{3}' \wedge_{\mathbf{1}} \mathbf{2}''\mathbf{3}'' \tag{3.72}$$

   is the unique factored expansion of r_{II}*. It is also the unique shortest expansion.*
3. *Double line. If* $\{\mathbf{1}, \mathbf{2}\} = \{\mathbf{1}', \mathbf{2}'\}$*, then* $\mathbf{12}$ *is called a* double line *in* r_{II}*. The following is the unique factored expansion, also the unique shortest one:*

$$\mathbf{123} \wedge \mathbf{123}' \wedge \mathbf{1}''\mathbf{2}''\mathbf{3}'' \wedge \mathbf{1}'''\mathbf{2}'''\mathbf{3}''' = [\mathbf{1233}']\mathbf{12} \wedge \mathbf{1}''\mathbf{2}''\mathbf{3}'' \wedge \mathbf{1}'''\mathbf{2}'''\mathbf{3}'''. \tag{3.73}$$

4. *Other cases. If r_{II} is not trivially zero, has neither triple point nor double line, then it has no factored expansion. The shortest expansions of r_{II} are* 2*-termed if and only if there exists a* 3*-tuple such that the intersections of this tuple with the other three tuples are respectively the three elements in the tuple. The corresponding expansion is*

$$\mathbf{123} \wedge \mathbf{12'3'} \wedge \mathbf{22''3''} \wedge \mathbf{32'''3'''} = [\mathbf{122'3'}][\mathbf{132'''3'''}][\mathbf{232''3''}] - [\mathbf{122''3''}][\mathbf{132'3'}][\mathbf{232'''3'''}]. \tag{3.74}$$

4. Factorization techniques in bracket computation

Cayley factorization, which changes a bracket polynomial to a Cayley expression, is the inverse procedure of Cayley expansion. While Cayley factorization techniques on bracket polynomials which are multilinear with respect to their vector variables are well developed (White, 1991), there is still no method to solve the general problem. In this section we propose several small algorithms based on the correspondences between the factored and nonfactored binomial Cayley expansions of the expressions p_I, p_{II} and p_{III}, assuming that all different points are generic ones. These algorithms prove to be sufficient for our task of automated theorem proving in both incidence and conic geometries, although more complicated algorithms based on multi-termed Cayley expansions are still possible.

4.1. Cayley factorization in two dimensions

First, two formulae can be derived from (3.2) for degree-2 Cayley factorization:

$$[\mathbf{A}_1\mathbf{A}_3\mathbf{A}_4][\mathbf{A}_2\mathbf{A}_5\mathbf{A}_6] - [\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4][\mathbf{A}_1\mathbf{A}_5\mathbf{A}_6] = \mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{A}_3\mathbf{A}_4 \wedge \mathbf{A}_5\mathbf{A}_6, \tag{4.1}$$

$$[\mathbf{CAB}_1][\mathbf{CDB}_2] - [\mathbf{CAB}_2][\mathbf{CDB}_1] = [\mathbf{CAD}][\mathbf{CB}_1\mathbf{B}_2]. \tag{4.2}$$

(4.2) will be implemented in the contraction algorithm in the next section. The following algorithm realizes (4.1).

Algorithm: Degree-2 Cayley factorization (4.1).
Input: A polynomial p composed of brackets and wedge products of type p_I, and already factored in the polynomial ring of these elements.
Output: A polynomial q of brackets and wedge products of type p_I.
Procedure. For every factor f of p, do the following.

Step 1. If f does not satisfy any of the following conditions, put it in q:
(1) f is a 2-termed bracket polynomial of degree two, and involves six points.
(2) The two terms are denoted by p_1, p_2. Their coefficients are ± 1.
Let $\mathbf{A}_1, \ldots, \mathbf{A}_6$ be the six points, and let $\mathbf{A}_i$ be the first point in p_1. Then $p_1 = \epsilon_1[\mathbf{A}_i\mathbf{A}_{j_1}\mathbf{A}_{j_2}][\mathbf{A}_{j_3}\mathbf{A}_{j_4}\mathbf{A}_{j_5}]$, $p_2 = \epsilon_2[\mathbf{A}_i\mathbf{A}_{k_1}\mathbf{A}_{k_2}][\mathbf{A}_{k_3}\mathbf{A}_{k_4}\mathbf{A}_{k_5}]$, where the ϵ's are the coefficients.
(3) $\epsilon_1\epsilon_2$ equals the sign of permutation of k_1 to k_5 relative to j_1 to j_5.

Step 2. Count the *bracket mates*, i.e. points in the same brackets, of each point in f. There are two points, denoted by $\mathbf{A}_1, \mathbf{A}_2$, each having four mates. Delete them from the bracket mates of the other four points. Then the four points each have one mate left, and thus form two pairs, denoted by $\mathbf{A}_3, \mathbf{A}_4$ and $\mathbf{A}_5, \mathbf{A}_6$ respectively.

Step 3. Now p_1 is of the form $\epsilon[\mathbf{A}_1\mathbf{A}_3\mathbf{A}_4][\mathbf{A}_2\mathbf{A}_5\mathbf{A}_6]$, where $\epsilon = \pm 1$. This defines an order in each of the pairs $\mathbf{A}_1\mathbf{A}_2$, $\mathbf{A}_3\mathbf{A}_4$ and $\mathbf{A}_5\mathbf{A}_6$. Put into q the following:

$$\epsilon\mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{A}_3\mathbf{A}_4 \wedge \mathbf{A}_5\mathbf{A}_6.$$

(3.12) and (3.15) provide the following formula for degree-3 Cayley factorization: let $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1}', \mathbf{2}', \mathbf{3}'$ be generic points. Then

$$\begin{aligned}&[\mathbf{A}_1\mathbf{A}_2\mathbf{B}_1][\mathbf{A}_2\mathbf{A}_3\mathbf{B}_2][\mathbf{A}_1\mathbf{A}_3\mathbf{B}_3] - [\mathbf{A}_1\mathbf{A}_2\mathbf{B}_2][\mathbf{A}_2\mathbf{A}_3\mathbf{B}_3][\mathbf{A}_1\mathbf{A}_3\mathbf{B}_1]\\ &\quad = [\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]\mathbf{A}_1\mathbf{B}_1 \wedge \mathbf{A}_2\mathbf{B}_2 \wedge \mathbf{A}_3\mathbf{B}_3.\end{aligned} \tag{4.3}$$

Algorithm: Degree-3 Cayley factorization (4.3).
Input: A polynomial p composed of brackets and wedge products of type p_I, and already factored in the polynomial ring of these elements.
Output: A polynomial q of brackets and wedge products of type p_I.
Procedure. For every factor f of p, do the following.

Step 1. If f does not satisfy any of the following conditions, put it in q:
(1) f is a 2-termed bracket polynomial of degree three, the coefficients are ± 1.
(2) f has six points, three of which are double points, denoted by $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$, the other three are single ones, denoted by $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$.

Step 2. Let the two terms be p_1, p_2. Then p_1 must be of the following form, which defines an order for the points: $p_1 = \epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{B}_1][\mathbf{A}_2\mathbf{A}_3\mathbf{B}_2][\mathbf{A}_1\mathbf{A}_3\mathbf{B}_3]$.
If $p_2 = -\epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{B}_2][\mathbf{A}_2\mathbf{A}_3\mathbf{B}_3][\mathbf{A}_1\mathbf{A}_3\mathbf{B}_1]$, put into q the following:

$$\epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]\mathbf{A}_1\mathbf{B}_1 \wedge \mathbf{A}_2\mathbf{B}_2 \wedge \mathbf{A}_3\mathbf{B}_3.$$

If $p_2 = -\epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{B}_3][\mathbf{A}_2\mathbf{A}_3\mathbf{B}_1][\mathbf{A}_1\mathbf{A}_3\mathbf{B}_2]$, put into q the following:

$$-\epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]\mathbf{A}_1\mathbf{B}_3 \wedge \mathbf{A}_2\mathbf{B}_1 \wedge \mathbf{A}_3\mathbf{B}_2.$$

In other cases, put f in q.

(3.7) provides a formula on degree-4 Cayley factorization:

$$\begin{aligned}&[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_5][\mathbf{A}_1\mathbf{A}_2\mathbf{A}_6][\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5][\mathbf{A}_3\mathbf{A}_4\mathbf{A}_6]\\ &\quad - [\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3][\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4][\mathbf{A}_5\mathbf{A}_6\mathbf{A}_3][\mathbf{A}_5\mathbf{A}_6\mathbf{A}_4]\\ &\quad\quad = (\mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{A}_3\mathbf{A}_5 \wedge \mathbf{A}_4\mathbf{A}_6)(\mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{A}_3\mathbf{A}_6 \wedge \mathbf{A}_4\mathbf{A}_5).\end{aligned} \tag{4.4}$$

Algorithm: Degree-4 Cayley factorization (4.4).
Input: A polynomial p composed of brackets and wedge products of type p_I, and already factored in the polynomial ring of these elements.
Output: A polynomial q of brackets and wedge products of type p_I.
Procedure. For every factor f of p, do the following.

Step 1. If f does not satisfy any of the following conditions, put it in q:
(1) f is a 2-termed bracket polynomial of degree four, the coefficients are ± 1.
(2) There are six points in f, each with degree 2.

Step 2. Count the number of bracket mates for each point in f. In the first term p_1, there are two points, denoted by $\mathbf{A}_5, \mathbf{A}_6$, each having four bracket mates, while the other four points each have three bracket mates. In the second term p_2, there are two other points, denoted by $\mathbf{A}_3, \mathbf{A}_4$, each having four bracket mates, while the other four points each have three bracket mates. The two points left are denoted by $\mathbf{A}_1, \mathbf{A}_2$.

Step 3. p_1 must be of the form $\epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_5][\mathbf{A}_1\mathbf{A}_2\mathbf{A}_6][\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5][\mathbf{A}_3\mathbf{A}_4\mathbf{A}_6]$. If $p_2 \neq -\epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3][\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4][\mathbf{A}_5\mathbf{A}_6\mathbf{A}_3][\mathbf{A}_5\mathbf{A}_6\mathbf{A}_4]$, put f in q, else put into q the following:

$$\epsilon(\mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{A}_3\mathbf{A}_5 \wedge \mathbf{A}_4\mathbf{A}_6)(\mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{A}_3\mathbf{A}_6 \wedge \mathbf{A}_4\mathbf{A}_5).$$

Remark. By theorems in Section 3, all the above Cayley factorizations are unique, and no more 2-termed factorization formulae can be derived from the Cayley expansions of p_I to p_{III} typed expressions.

The following is a simple algorithm *Cayley combination*, which combines a polynomial of brackets and p_I-typed wedge products using the Cayley factorization algorithms. Although lacking generality, it can finish all the Cayley factorization tasks we meet in theorem proving, see Li and Wu (2003).

Algorithm: Cayley combination.
Input: A polynomial p composed of brackets and p_I- typed wedge products. Assume that p is already factored in the polynomial ring of these elements.
Output: A polynomial q of brackets and wedge products of type p_I.
Procedure: While p is not empty, for every factor f of p, do the following.

Step 1. If f is a monomial, move it to q.

Step 2. For every pair of terms in f, do degree-2 to degree-4 Cayley factorizations. Every time after a successful Cayley factorization, factor f in the polynomial ring of brackets and wedge products. If f becomes factored, replace it by the factors, and go back to the beginning of the Procedure.

Step 3. Move f to q.

Example 4.1. Let $\mathbf{1}, \ldots, \mathbf{6}, \mathbf{1}', \mathbf{2}'$ be generic points. Let

$$\begin{aligned} p = {} & [\mathbf{1}'\mathbf{12}][\mathbf{2}'\mathbf{23}][\mathbf{3}'\mathbf{13}][\mathbf{456}] - [\mathbf{2}'\mathbf{12}][\mathbf{3}'\mathbf{23}][\mathbf{1}'\mathbf{13}][\mathbf{456}] \\ & - [\mathbf{11}'\mathbf{2}'][\mathbf{233}'][\mathbf{124}][\mathbf{356}] + [\mathbf{121}'][\mathbf{32}'\mathbf{3}'][\mathbf{124}][\mathbf{356}]. \end{aligned}$$

There are only two pairs of terms having Cayley factorizations. The first two terms have the following degree-3 Cayley factorization:

$$[\mathbf{1}'\mathbf{12}][\mathbf{2}'\mathbf{23}][\mathbf{3}'\mathbf{13}] - [\mathbf{2}'\mathbf{12}][\mathbf{3}'\mathbf{23}][\mathbf{1}'\mathbf{13}] = [\mathbf{123}]\mathbf{11}' \wedge \mathbf{22}' \wedge \mathbf{33}'.$$

The last two terms have the following degree-2 Cayley factorization:

$$-[\mathbf{11}'\mathbf{2}'][\mathbf{233}'] + [\mathbf{121}'][\mathbf{32}'\mathbf{3}'] = -\mathbf{11}' \wedge \mathbf{22}' \wedge \mathbf{33}'.$$

So

$$p = ([\mathbf{123}][\mathbf{456}] - [\mathbf{124}][\mathbf{356}])\mathbf{11}' \wedge \mathbf{22}' \wedge \mathbf{33}' = -(\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56})(\mathbf{11}' \wedge \mathbf{22}' \wedge \mathbf{33}')$$

by another degree-2 Cayley factorization.

4.2. Cayley factorization in higher dimensions

When the brackets are taken as Mourrain ones, or the 3-elemented brackets are supplemented with $n-3$ common vectors to become brackets in $(n-1)$-dimensional projective space, then obviously the factorizations in the previous subsection are still valid.

In this subsection, we consider a generalization of the degree-2 factorization (4.1) in $(n-1)$-dimensional projective space. Let

$$B = A_{r_1} \wedge A_{r_2} \wedge A_{r_3}, \tag{4.5}$$

where r_1, r_2, r_3 are integers, $A_{r_i} = \mathbf{A}_{i1} \cdots \mathbf{A}_{ir_i}$, and the set of the $\mathbf{A}$'s is a set of generic points. Assume that $r_1+r_2+r_3 = 2n$, $1 < r_i < n$, and $r_i + r_j > n$ for any $1 \le i < j \le 3$.

A *Cayley expansion* of B by distributing A_{r_1} is the following:

$$A_{r_1} \wedge A_{r_2} \wedge A_{r_3} = \sum_{C_{n-r_2} \subseteq A_{r_1}} \mathrm{sign}(C_{n-r_2}, C^*_{n-r_3})[C_{n-r_2} A_{r_2}][C^*_{n-r_3} A_{r_3}] \tag{4.6}$$

where C_{n-r_2} is a subsequence of $n-r_2$ elements in A_{r_1}, and $C^*_{n-r_3}$ is the remainder of C_{n-r_2} in A_{r_1}, which is also a subsequence of A_{r_1}. There are three such ways to expand B.

Proposition 4.1. *Let $r_1+r_2+r_3 = 2n$, $1 < r_i < n$, and $r_i+r_j > n$ for any $1 \le i < j \le 3$. Let $0 \le t \le \min(r_2, r_3)$. Let $A'_{r_1}, A'_{r_2-t}, A'_{r_3-t}, D'_t$ be four sequences of generic points, whose pairwise intersections are all empty. Then*

$$\begin{aligned} &\sum_{C_{n-r_2} \subseteq A'_{r_1}} \mathrm{sign}(C_{n-r_2}, C^*_{n-r_3})[C_{n-r_2} A'_{r_2-t} D'_t][C^*_{n-r_3} A'_{r_3-t} D'_t] \\ &\quad = A'_{r_1} \wedge (A'_{r_2-t} D'_t) \wedge (A'_{r_3-t} D'_t). \end{aligned} \tag{4.7}$$

The factorization is unique in the sense that if the left side of (4.7) equals $A_{s_1} \wedge A_{s_2} \wedge A_{s_3}$, where the A's are sequences of vectors, then by the following transformations it can be changed into $A'_{r_1} \wedge (A'_{r_2-t} D'_t) \wedge (A'_{r_3-t} D'_t)$: let $E_{t_1}, E_{t_2}, E_{t_3}$ be three sequences of t_1, t_2, t_3 vectors respectively, where the t's are nonnegative integers such that $t_1+t_2+t_3 = 2n-4$, then

$$\begin{aligned} &\mathbf{12}E_{t_1} \wedge \mathbf{1}E_{t_2} \wedge \mathbf{2}E_{t_3} = (-1)^{n-t_3}\mathbf{12}E_{t_1} \wedge \mathbf{12}E_{t_2} \wedge E_{t_3}, \\ &\mathbf{12}E_{t_1} \wedge \mathbf{12}E_{t_2} \wedge E_{t_3} = \mathbf{12}E_{t_1} \wedge E_{t_2} \wedge \mathbf{12}E_{t_3}. \end{aligned} \tag{4.8}$$

When $t = n - r_1$, (4.7) can be simplified as follows:

$$\begin{aligned} &\sum_{C_{n-r_2} \subseteq A'_{r_1}} \mathrm{sign}(C_{n-r_2}, C^*_{n-r_3})[C_{n-r_2} A'_{n-r_3} D'_{n-r_1}][C^*_{n-r_3} A'_{n-r_2} D'_{n-r_1}] \\ &\quad = [A'_{r_1} D'_{n-r_1}][A'_{n-r_3} A'_{n-r_2} D'_{n-r_1}]. \end{aligned} \tag{4.9}$$

Proof. First we prove (4.8). The first equality comes from the observation that when both sides are expanded by distributing $\mathbf{12}E_{t_1}$, the results are the same:

$$\sum_{F_{n-2-t_2}\subseteq E_{t_1}} \mathrm{sign}(\mathbf{2}F_{n-2-t_2}, \mathbf{1}F^*_{n-2-t_3})[\mathbf{2}F_{n-2-t_2}\mathbf{1}E_{t_2}][\mathbf{1}F^*_{n-2-t_3}\mathbf{2}E_{t_3}] = (-1)^{n-t_3}$$
$$\times \sum_{F_{n-2-t_2}\subseteq E_{t_1}} \mathrm{sign}(F_{n-2-t_2}, F^*_{n-2-t_3})[\mathbf{12}F_{n-2-t_2}E_{t_2}][\mathbf{12}F^*_{n-2-t_3}E_{t_3}].$$

Similarly, when both sides of the second equality are expanded by distributing $\mathbf{12}E_{t_1}$, the results are the same.

By the transformations (4.8), $A_{s_1} \wedge A_{s_2} \wedge A_{s_3}$ can be changed into $p'' = A''_{t_1} \wedge (A''_{t_2-s}D''_s) \wedge (A''_{t_3-s}D''_s)$, where $A''_{t_1}, A''_{t_2-s}, A''_{t_3-s}, D''_s$ are sequences of generic points whose pairwise intersections are empty. Below we compare p'' with $p' = A'_{r_1} \wedge (A'_{r_2-t}D'_t) \wedge (A'_{r_3-t}D'_t)$:

(1) By homogeneity, the sets of points in the two wedge products are the same.
(2) $s = t$ and $D''_s = D'_t$. The reason is that p' (or p'') is quadratic only with respect to points in D'_t (or D''_s).
(3) p' (or p'') is antisymmetric with respect to any two points in A'_{r_i-t} (or A''_{t_i-s}). Then given $1 \le i \le 3$, there exists $1 \le j \le 3$ such that $A'_{r_i-t} = A''_{t_j-s}$.

Therefore, the components of p'' and p' are identical. To prove (4.9), we count the numbers of terms in different expansions of $B = A'_{r_1} \wedge (A'_{r_2-t}D'_t) \wedge (A'_{r_3-t}D'_t)$. Distributing A'_{r_1}, we get $C^{n-r_2}_{r_1} > 1$ terms; distributing $A'_{r_2-t}D'_t$, we get $C^{n-r_3}_{r_2-t}$ terms; distributing $A'_{r_3-t}D'_t$, we get $C^{n-r_2}_{r_3-t}$ terms. So B has monomial expansion if and only if $r_2 - t = n - r_3$ or $r_3 - t = n - r_2$, i.e. $t = n - r_1$. □

5. Simplification techniques in bracket computation

While the GP relations can be used in Cramer's rules for elimination and coordinatization (Mourrain, unpublished work), and in the straightening algorithm for normalization (Sturmfels and White, 1989), they can be used directly in the procedure of bracket polynomial computation, to reduce the number of terms by finding a shorter but equal polynomial. This idea leads to a series of powerful techniques for simplifying bracket computation. We start with an analysis of the structures of GP polynomials.

5.1. Three-termed Grassmann–Plücker polynomials

Let $\dim(\mathcal{V}) = n$. Then the numbers of terms of GP polynomials range from 3 to $n + 1$. Any $(r + 1)$-termed GP polynomial g_{r+1} is of the following form:

$$g_{r+1} = \sum_{i=1}^{r+1} (-1)^{i+1}[\mathbf{C}_1 \cdots \mathbf{C}_{n-r}\mathbf{A}_1 \cdots \mathbf{A}_{r-1}\mathbf{B}_i] \times [\mathbf{C}_1 \cdots \mathbf{C}_{n-r}\mathbf{B}_1 \cdots \check{\mathbf{B}}_i \cdots \mathbf{B}_{r+1}], \tag{5.1}$$

where $\mathbf{A}_1, \ldots, \mathbf{A}_{r-1}, \mathbf{B}_1, \ldots, \mathbf{B}_{r+1}, \mathbf{C}_1, \ldots, \mathbf{C}_{n-r}$ are generic vectors in $\mathcal{V}$.

The following proposition indicates that 3-termed GP polynomials are theoretically sufficient for verifying all kinds of bracket identities.

Proposition 5.1. *Any polynomial in* $\mathcal{I}_{n,m}$*, when multiplied by a suitable bracket monomial, is in the ideal generated by* 3*-termed polynomials in* $\mathcal{GP}_{n,m}$.

Proof. We use induction on the number of terms r of GP polynomials. The case $r = 3$ is trivial. In (5.1), we use the following notations: $\gamma = \mathbf{C}_1 \cdots \mathbf{C}_{n-r}$, $\alpha = \gamma \mathbf{A}_1 \cdots \mathbf{A}_{r-2}$, $\beta_i = \mathbf{B}_1 \cdots \check{\mathbf{B}}_i \cdots \mathbf{B}_{r+1}$. Then

$$g_{r+1} = \sum_{i=1}^{r+1} (-1)^{i+1} [\alpha \mathbf{A}_{r-1} \mathbf{B}_i][\gamma \beta_i].$$

Let $g_r(\mathbf{B}_1) = \sum_{i=1}^{r+1} (-1)^{i+1} [\alpha \mathbf{B}_1 \mathbf{B}_i][\gamma \beta_i]$, $g_r(\mathbf{B}_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} [\alpha \mathbf{B}_i \mathbf{B}_{r+1}][\gamma \beta_i]$. They are both r-termed GP polynomials. It can be verified that

$$\begin{aligned} [\alpha \mathbf{B}_1 \mathbf{B}_{r+1}] g_{r+1} = {} & [\alpha \mathbf{A}_{r-1} \mathbf{B}_{r+1}] g_r(\mathbf{B}_1) + [\alpha \mathbf{A}_{r-1} \mathbf{B}_1] g_r(\mathbf{B}_{r+1}) \\ & + \sum_{i=2}^{r} (-1)^i [\alpha \beta_i] g_3(\mathbf{B}_i), \end{aligned} \tag{5.2}$$

where $g_3(\mathbf{B}_i) = [\alpha \mathbf{A}_{r-1} \mathbf{B}_1][\alpha \mathbf{B}_i \mathbf{B}_{r+1}] - [\alpha \mathbf{A}_{r-1} \mathbf{B}_i][\alpha \mathbf{B}_1 \mathbf{B}_{r+1}] + [\alpha \mathbf{A}_{r-1} \mathbf{B}_{r+1}][\alpha \mathbf{B}_1 \mathbf{B}_i]$ is a 3-termed GP polynomial. □

In practice, however, using exclusively 3-termed GP relations is very inefficient.

5.2. Contraction

Given any r-termed GP polynomial g_r, we can always divide it into two parts $g_{r,r'} + g_{r,r-r'}$ with r', $r - r'$ terms respectively, where $r' > 1$. The transformation

$$g_{r,r'} = -g_{r,r-r'} \tag{5.3}$$

is called a GP *transformation*. When applying it to a polynomial p, by the computation rules of brackets, if the number of terms of p is decreased, we say p is *contractible*, and p is *contracted* when it is replaced by the new result. Notice that it is not necessary for $r' > r/2$.

A natural question is, given the left side of (5.3), is the right side unique? The answer is no. By (5.1), the general form of an r'-termed part of an r-termed GP polynomial is

$$\begin{aligned} g_{r,r'} = {} & \sum_{i=1}^{r'} (-1)^{i+1} [\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1} \mathbf{A}_1 \cdots \mathbf{A}_{r-2} \mathbf{B}'_i] \\ & \times [\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1} \mathbf{D}_1 \cdots \mathbf{D}_{r-r'} \mathbf{B}'_1 \cdots \check{\mathbf{B}}'_i \cdots \mathbf{B}'_{r'}], \end{aligned} \tag{5.4}$$

where $\mathbf{B}'_1, \ldots, \mathbf{B}'_{r'}, \mathbf{D}_1, \ldots, \mathbf{D}_{r-r'}$ are the $\mathbf{B}$'s in (5.1). Comparing (5.4) with the left side of (4.7), we find that if $r' < r - 1$, then by the correspondences $r_1 = r'$, $r_2 = n - 1$, $r_3 = n - r' + 1$ and $t = n - r + 1$, $g_{r,r'}$ has the following Cayley factorization:

$$\begin{aligned} g_{r,r'} = {} & (-1)^{rr'} (\mathbf{B}'_1 \cdots \mathbf{B}'_{r'}) \wedge (\mathbf{A}_1 \cdots \mathbf{A}_{r-2} \mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}) \\ & \wedge (\mathbf{D}_1 \cdots \mathbf{D}_{r-r'} \mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}). \end{aligned} \tag{5.5}$$

If $r' = r - 1$, then

$$g_{r-1,r} = [\mathbf{B}'_1 \cdots \mathbf{B}'_{r-1}\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}][\mathbf{A}_1 \cdots \mathbf{A}_{r-2}\mathbf{D}_1\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}]. \tag{5.6}$$

If $r' = r$, then $g_{r,r} = 0$. Since (5.5) has three expansions, the right side of (5.3) has two possibilities if $r' < r - 1$, and is unique otherwise.

In practice, it is sufficient to choose $r' = 2$. By the above arguments, a contraction can be taken as a Cayley factorization followed by a different Cayley expansion in the case of (5.5), and just a Cayley factorization in other cases. The following is an algorithm realizing contractions in n-dimensional vector space $\mathcal{V}$. The formula is

$$\begin{aligned}
&[\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}\mathbf{A}_1 \cdots \mathbf{A}_{r-2}\mathbf{B}'_1][\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}\mathbf{D}_1 \cdots \mathbf{D}_{r-2}\mathbf{B}'_2]\\
&\quad - [\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}\mathbf{A}_1 \cdots \mathbf{A}_{r-2}\mathbf{B}'_2][\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}\mathbf{D}_1 \cdots \mathbf{D}_{r-2}\mathbf{B}'_1]\\
&= \begin{cases}
\sum_{i=1}^{r-2}(-1)^{i+1}[\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}\mathbf{B}'_1\mathbf{B}'_2\mathbf{A}_1 \cdots \check{\mathbf{A}}_i \cdots \mathbf{A}_{r-2}] & \\
\times [\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}\mathbf{A}_i\mathbf{D}_1 \cdots \mathbf{D}_{r-2}] & \\
= \sum_{i=1}^{r-2}(-1)^{i}[\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}\mathbf{B}'_1\mathbf{B}'_2\mathbf{D}_1 \cdots \check{\mathbf{D}}_i \cdots \mathbf{D}_{r-2}] & \\
\times [\mathbf{C}_1 \cdots \mathbf{C}_{n-r+1}\mathbf{D}_i\mathbf{A}_1 \cdots \mathbf{A}_{r-2}], & \\
& \text{if } r > 3;\\
[\mathbf{C}_1 \cdots \mathbf{C}_{n-2}\mathbf{B}'_1\mathbf{B}'_2][\mathbf{C}_1 \cdots \mathbf{C}_{n-2}\mathbf{A}_1\mathbf{D}_1], & \text{if } r = 3.
\end{cases}
\end{aligned} \tag{5.7}$$

Algorithm: Contraction.

Input: A bracket polynomial p of degree at least two. Assume that p is already factored in the polynomial ring of brackets.

Output: A bracket polynomial q.

Procedure: Move monomial factors of p to q. While p is not empty, for every factor f of p, do the following.

Step 1 (Contraction on the same level).
Set $g = 0$. For every pair of terms $p_1 + p_2$ in f, do the following:

1.1. Let $p_1 = cd_1$, $p_2 = cd_2$, where c is their common factors. If d_1, d_2 are not each composed of two brackets with coefficient ± 1, skip to the next pair of terms.

1.2. Count the degrees of the points in d_1. Let $\gamma = \mathbf{C}_1, \ldots, \mathbf{C}_{n-r+1}$ be the double points. By this we obtain the number r.

1.3. If $r = 3$, then there are only four single points in $d_1 + d_2$. Do the following:
(1) Fix a single point $\mathbf{A}_1$. Its single-point bracket mates in d_1, d_2 are denoted by $\mathbf{B}'_1$, $\mathbf{B}'_2$ respectively. The fourth single point is denoted by $\mathbf{D}_1$.
(2) d_1 must be of the form $\epsilon[\gamma\mathbf{A}_1\mathbf{B}'_1][\gamma\mathbf{D}_1\mathbf{B}'_2]$, where $\epsilon = \pm 1$. If $d_2 \neq -\epsilon[\gamma\mathbf{A}_1\mathbf{B}'_2][\gamma\mathbf{D}_1\mathbf{B}'_1]$, skip to the next pair of terms.
(3) Set $g = g + \epsilon c[\gamma\mathbf{B}'_1\mathbf{B}'_2][\gamma\mathbf{A}_1\mathbf{D}_1]$. Remove the pair of terms from f.

1.4. If $r > 3$, do the following:
(1) Count the single-point bracket mates of the single points in $d_1 + d_2$. There are two points, denoted by $\mathbf{B}'_1, \mathbf{B}'_2$, each with $2r - 4$ such mates, while the other $2r - 4$ points each have $r - 1$ such mates.

If this is not the case, skip to the next pair of terms.
(2) d_1 must be of the form $\epsilon[\gamma \mathbf{A}_1 \cdots \mathbf{A}_{r-2}\mathbf{B}'_1][\gamma \mathbf{D}_1 \cdots \mathbf{D}_{r-2}\mathbf{B}'_2]$. The $2r-4$ points are thus separated into two groups: $\mathbf{A}_1, \ldots, \mathbf{A}_{r-2}$ and $\mathbf{D}_1, \ldots, \mathbf{D}_{r-2}$. If $d_2 \neq -\epsilon[\gamma \mathbf{D}_1 \cdots \mathbf{D}_{r-2}\mathbf{B}'_1][\gamma \mathbf{A}_1 \cdots \mathbf{A}_{r-2}\mathbf{B}'_2]$, skip to the next pair of terms.
(3) Set p'_1, p'_2 to be the first two expressions in (5.7). If when substituting the pair of terms with $\epsilon c p'_i$ after expansion for some $i = 1, 2$, the number of terms in f is reduced, then do the substitution, keep the unchanged terms in f and move all others to g.

Step 2. (Further contraction). Set $f = f + g$.
If $f = 0$ then return $q = 0$, else if f is a monomial or no contraction occurs in Step 1, then move f to q, else go back to Step 1.

Example 5.1 (From Example 7.6 in Section 7). Let $p = [\mathbf{135}][\mathbf{2345}] - [\mathbf{235}][\mathbf{1345}]$, where point $\mathbf{5}$ is on plane $\mathbf{123}$.

For Mourrain brackets, we simply add dummy vectors $\mathbf{U}$'s to make them equal in length. Then $p = [\mathbf{135U}_1][\mathbf{2345}] - [\mathbf{235U}_1][\mathbf{1345}]$. The points with their degrees are $\mathbf{1}^1$, $\mathbf{2}^1$, $\mathbf{3}^2$, $\mathbf{4}^1$, $\mathbf{5}^2$, $\mathbf{U}_1^1$. So $\gamma = \mathbf{35}$ and $r = 3$.

Fix single point $\mathbf{A}_1 = \mathbf{1}$. Its single-point bracket mates are $\mathbf{B}'_1 = \mathbf{U}_1$ and $\mathbf{B}'_2 = \mathbf{4}$. The fourth single point is $\mathbf{D}_1 = \mathbf{2}$. The first term of p equals $-[\mathbf{351U}_1][\mathbf{3524}]$ and the second term equals $[\mathbf{3514}][\mathbf{352U}_1]$, so $p = -[\mathbf{35U}_1\mathbf{4}][\mathbf{3512}] = 0$ by the computation rule of brackets.

5.3. Level contraction

Let $g_{r,r'} = -g_{r,r-r'}$ be a GP transformation of a bracket polynomial p. Let the corresponding terms in p be the expansion of the multiplication $\lambda_{r,r'} g_{r,r'}$, where $\lambda_{r,r'}$ is a bracket polynomial. The *range* of the GP transformation is the set of terms up to scale in the expansion of the multiplication $\lambda_{r,r'}(g_{r,r'} + g_{r,r-r'})$, where only the bracket computation rules are applied. A group of GP transformations of p are said to be *on the same level* if no two transformations have intersecting ranges.

A *level contraction* of a noncontractible bracket polynomial is composed of one or several GP transformations and one or several successive contractions, the latter being on the same level, such that the number of terms of the polynomial is decreased after the transformations.

Algorithm: Level contraction.

Input: A bracket polynomial p of degree at least two. Assume that p is neither factorable in the polynomial ring of brackets, nor contractible.

Output: p after level contractions.

Stage one: single level transformation. For every pair of terms in p, use Step 1.4 in the contraction algorithm to detect if it is GP transformable, and skip to the next pair if it is not.

Let $p_1 + p_2$ be a GP transformable pair, with two transformation results p_1' and p_2'. If when p_i' is substituted into p, the result can be contracted to less terms than the original p, then carry out the transformations, return p.

Stage two: combined level transformations. It requires that p has at least two GP transformable pairs.

For the GP transformable pairs in p, combine their identity transformations and GP transformations in different ways. Each combination should contain at least two GP transformations. If there is any combination that makes the transformation result contracted to less terms than the original p, carry it out and return p.

Example 5.2 (From Example 7.4 in Li and Wu, 2003). Let

$$\begin{aligned}
p = {} & [\mathbf{124}][\mathbf{125}][\mathbf{127}][\mathbf{136}][\mathbf{236}][\mathbf{357}][\mathbf{457}] \\
& + [\mathbf{125}][\mathbf{126}][\mathbf{127}][\mathbf{134}][\mathbf{234}][\mathbf{357}][\mathbf{567}] \\
& - [\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{234}][\mathbf{257}][\mathbf{567}] \\
& - [\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{236}][\mathbf{257}][\mathbf{457}] \\
& + [\mathbf{124}][\mathbf{126}][\mathbf{134}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{567}] \\
& + [\mathbf{124}][\mathbf{126}][\mathbf{136}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{457}].
\end{aligned}$$

There are only two pairs of terms in p with five common bracket factors. Each pair is GP transformable.

(1) Terms 3 and 4:

$$\begin{aligned}
& -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{257}]([\mathbf{234}][\mathbf{567}] + [\mathbf{236}][\mathbf{457}]) \\
& \overset{GP}{=} -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{235}][\mathbf{257}][\mathbf{467}] \\
& \qquad -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{237}][\mathbf{257}][\mathbf{456}] \\
& \overset{GP}{=} -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{257}][\mathbf{246}][\mathbf{357}] \\
& \qquad -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{257}]^2[\mathbf{346}].
\end{aligned} \tag{5.8}$$

None of the four new terms has five common bracket factors with any of the remaining terms in p. So this GP transformable pair is not contractible, nor does it induce any contraction.

(2) Terms 5 and 6:

$$\begin{aligned}
& [\mathbf{124}][\mathbf{126}][\mathbf{157}][\mathbf{235}][\mathbf{237}]([\mathbf{134}][\mathbf{567}] + [\mathbf{136}][\mathbf{457}]) \\
& \overset{GP}{=} [\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{467}] \\
& \qquad +[\mathbf{124}][\mathbf{126}][\mathbf{137}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{456}] \\
& \overset{GP}{=} [\mathbf{124}][\mathbf{126}][\mathbf{146}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{357}] \\
& \qquad -[\mathbf{124}][\mathbf{126}][\mathbf{157}]^2[\mathbf{235}][\mathbf{237}][\mathbf{346}].
\end{aligned} \tag{5.9}$$

Similarly, this GP transformable pair is neither contractible nor induces any contraction.

Now we do combination to the GP transformations. For each pair of transformations, we only need to consider the four combinations of the four newly produced terms. The conclusion is that the first transformations in (5.8) and (5.9) produce two contractible pairs:

$$-[\mathbf{137}][\mathbf{257}]+[\mathbf{157}][\mathbf{237}]=-[\mathbf{127}][\mathbf{357}],\ -[\mathbf{135}][\mathbf{257}]+[\mathbf{157}][\mathbf{235}] = -[\mathbf{125}][\mathbf{357}].$$

So

$$\begin{aligned} p = [\mathbf{357}]\{&[\mathbf{124}][\mathbf{125}][\mathbf{127}][\mathbf{136}][\mathbf{236}][\mathbf{457}]+[\mathbf{125}][\mathbf{126}][\mathbf{127}][\mathbf{134}][\mathbf{234}][\mathbf{567}] \\ &-[\mathbf{124}][\mathbf{125}][\mathbf{126}][\mathbf{137}][\mathbf{237}][\mathbf{456}] \\ &-[\mathbf{124}][\mathbf{126}][\mathbf{127}][\mathbf{135}][\mathbf{235}][\mathbf{467}]\}, \end{aligned} \tag{5.10}$$

which has two terms less than the original p. The other three pairs of transformations cannot produce any pair of terms with five common bracket factors.

5.4. Strong contraction

Let g_r be an r-termed GP polynomial. Let t be a term in g_r, and let $g_r(t) = g_r - t$. The transformation

$$t = -g_r(t) \tag{5.11}$$

is called an *explosion*. Given t, if $r = 3$, the right side is unique; if $r > 3$, the right side has $2r - 2$ different possibilities.

A *strong contraction* of a bracket polynomial is an explosion followed by several contractions on the same level, such that the number of terms of the polynomial is decreased after the transformations.

Proposition 5.2. *Let p be a bracket polynomial whose different points do not satisfy any incidence constraint.*

1. *If p has no GP transformation but has a strong contraction induced by an r-termed GP polynomial, then p has at least r terms, and the strong contraction has $r - 1$ successive contractions on the same level. Each contraction is from a term generated by the explosion and an original term in p to a monomial.*
2. *If every pair of terms in p has at least $i \geq 3$ brackets left after the removal of their common bracket factors, then p has no GP transformation.* (1) *If $i \geq 5$, then p has no strong contraction.* (2) *If $i = 4$, then in each contractible pair during a strong contraction, the two brackets generated by the explosion are common factors.* (3) *If $i = 3$, then in each contractible pair during a strong contraction, at least one of the two brackets generated by the explosion is a common factor.*

The following algorithm realizes strong contractions for $n = 3$. The formulae are

$$\begin{aligned} [\mathbf{123}][\mathbf{456}] &= [\mathbf{124}][\mathbf{356}]-[\mathbf{125}][\mathbf{346}]+[\mathbf{126}][\mathbf{345}] \\ &= -[\mathbf{134}][\mathbf{256}]+[\mathbf{135}][\mathbf{246}]-[\mathbf{136}][\mathbf{245}] \\ &= [\mathbf{234}][\mathbf{156}]-[\mathbf{235}][\mathbf{146}]+[\mathbf{236}][\mathbf{145}] \\ &= [\mathbf{145}][\mathbf{236}]-[\mathbf{245}][\mathbf{136}]+[\mathbf{345}][\mathbf{126}] \end{aligned}$$

$$= [\mathbf{146}][\mathbf{235}] - [\mathbf{246}][\mathbf{135}] + [\mathbf{346}][\mathbf{125}]$$
$$= [\mathbf{156}][\mathbf{234}] - [\mathbf{256}][\mathbf{134}] + [\mathbf{356}][\mathbf{124}] \tag{5.12}$$

and

$$[\mathbf{123}][\mathbf{145}] = [\mathbf{124}][\mathbf{135}] - [\mathbf{125}][\mathbf{134}]. \tag{5.13}$$

Algorithm: Strong contraction for $n = 3$.

Input: A bracket polynomial p of at least three terms and five points. Assume that p is not factorable in the polynomial ring of brackets, and its points do not satisfy any incidence constraint.

Output: p after strong contraction, or p itself.

Step 1. For every pair of terms in p, compute the degree of the remaining polynomial after the removal of their common bracket factors. Let i be the lowest of such degrees. If $i \geq 5$ then there is no strong contraction.

Step 2. For every bracket in p, compute its total degree, which is the sum of the degrees of its points in p. The total degree defines an order among the brackets in p.

Step 3 (Explosion by 4-termed GP polynomial). If p has at least four terms and involves at least six points, then start from the bracket with the lowest total degree, say $[\mathbf{123}]$, do the following:

Let $\mathcal{S}$ be the points in p other than $\mathbf{1}, \mathbf{2}, \mathbf{3}$. Let P be the set of terms in p containing $[\mathbf{123}]$. For any element $p' \in P$, start from the bracket $[\mathbf{456}]$ in p' with the lowest total degree, where $\{\mathbf{4}, \mathbf{5}, \mathbf{6}\} \subseteq \mathcal{S}$, by letting $p' = [\mathbf{123}][\mathbf{456}]p''$, do the following.

Case 1. If $i < 3$, let q_1 to q_6 be the expanded form of p'' multiplied by the six polynomials in (5.12) respectively. For each term in q_j, check if there is any term in $p - p'$ that can form a contractible pair with it, using Step 1.3 in the contraction algorithm. If this is true for each term in a q_j, replace p' and the three terms involved in the contractions by the contraction results, exit.

Case 2. If $i = 3$, find in (5.12) such polynomials that each term of the polynomial has a bracket in a different term of $p - p'$. This establishes a set of correspondences between the three terms of the polynomials in (5.12) multiplied by p'', and the 3-tuples of terms in $p - p'$. For each correspondence, check if the corresponding pairs are all contractible, using Step 1.3 in the contraction algorithm, and if so, replace p' and the three terms involved in the contractions by the contraction results, exit.

Case 3. If $i = 4$, find in (5.12) such polynomials that each term of the polynomial is in a different term of $p - p'$. This establishes a set of correspondences between the three terms of the polynomials in (5.12) multiplied by p'', and the 3-tuples of terms in $p - p'$. For each correspondence, check if the corresponding pairs are all contractible, using Step 1.3 in the contraction algorithm, and if so, replace p' and the three terms involved in the contractions by the contraction results, exit.

Step 4 (Explosion by 3-termed GP polynomial). Start from the bracket q with the lowest total degree, do the following:

Let $\mathcal{S}$ be the points in p but not in q. Let P be the set of terms in p containing q. For any element $p' \in P$, start from the bracket $[\mathbf{145}]$ in p' with the lowest total degree, where $\{\mathbf{4}, \mathbf{5}\} \subseteq \mathcal{S}$ and $\mathbf{1} \in q$, by letting $q = [\mathbf{123}]$ and $p' = [\mathbf{123}][\mathbf{145}]p''$, do the following.

For each term in $p''[\mathbf{124}][\mathbf{135}] - p''[\mathbf{125}][\mathbf{134}]$, check if there is any term in $p - p'$ that can form a contractible pair with it, using Step 1.3 in the contraction algorithm. If this is true for both terms, replace p' and the two terms involved in the contractions by the contraction results, exit.

Example 5.3 (From Example 7.4 in Li and Wu, 2003). Let p be the polynomial factor in (5.10). The lowest degree of pairwise terms with common bracket factors removed is $i = 4$. The points with their degrees are $\mathbf{1}^4, \mathbf{2}^4, \mathbf{3}^2, \mathbf{4}^2, \mathbf{5}^2, \mathbf{6}^2, \mathbf{7}^2$. The brackets with their total degrees are $[\mathbf{12j}]^{10}$, $[\mathbf{13j}]^8$, $[\mathbf{23j}]^8$ for $4 \leq j \leq 7$, and $[\mathbf{jkl}]^6$ for $4 \leq j < k < l \leq 7$.

We start from any bracket with total degree 6, for example $[\mathbf{567}]$. Then $\mathcal{S} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$. The second term p' is the only one that contains $[\mathbf{567}]$. There are two brackets in p' with total degree 8 and formed by points in $\mathcal{S}$: $[\mathbf{134}]$, $[\mathbf{234}]$. Start from any one, say $[\mathbf{134}]$. Then $p'' = [\mathbf{125}][\mathbf{126}][\mathbf{127}][\mathbf{234}]$.

The corresponding six explosions are

$$\begin{aligned}
[\mathbf{134}][\mathbf{567}] &= [\mathbf{135}][\mathbf{467}] - [\mathbf{136}][\mathbf{457}] + [\mathbf{137}][\mathbf{456}] \\
&= -[\mathbf{145}][\mathbf{367}] + [\mathbf{146}][\mathbf{357}] - [\mathbf{147}][\mathbf{356}] \\
&= [\mathbf{345}][\mathbf{167}] - [\mathbf{346}][\mathbf{157}] + [\mathbf{347}][\mathbf{156}] \\
&= [\mathbf{167}][\mathbf{345}] - [\mathbf{367}][\mathbf{145}] + [\mathbf{467}][\mathbf{135}] \\
&= -[\mathbf{157}][\mathbf{346}] + [\mathbf{357}][\mathbf{146}] - [\mathbf{457}][\mathbf{136}] \\
&= [\mathbf{156}][\mathbf{347}] - [\mathbf{356}][\mathbf{147}] + [\mathbf{456}][\mathbf{137}].
\end{aligned} \tag{5.14}$$

Only the first explosion has its three terms in the three terms of $p - p'$ respectively. The correspondence is

$$\begin{array}{rcl}
p''[\mathbf{135}][\mathbf{467}] & \Longleftrightarrow & -[\mathbf{124}][\mathbf{126}][\mathbf{127}][\mathbf{135}][\mathbf{235}][\mathbf{467}], \\
-p''[\mathbf{136}][\mathbf{457}] & \Longleftrightarrow & [\mathbf{124}][\mathbf{125}][\mathbf{127}][\mathbf{136}][\mathbf{236}][\mathbf{457}], \\
p''[\mathbf{137}][\mathbf{456}] & \Longleftrightarrow & -[\mathbf{124}][\mathbf{125}][\mathbf{126}][\mathbf{137}][\mathbf{237}][\mathbf{456}].
\end{array}$$

The three corresponding pairs are all contractible. The contractions are

$$\begin{array}{rcl}
[\mathbf{125}][\mathbf{234}] - [\mathbf{124}][\mathbf{235}] & = & -[\mathbf{123}][\mathbf{245}] \\
[\mathbf{124}][\mathbf{236}] - [\mathbf{126}][\mathbf{234}] & = & [\mathbf{123}][\mathbf{246}] \\
[\mathbf{127}][\mathbf{234}] - [\mathbf{124}][\mathbf{237}] & = & -[\mathbf{123}][\mathbf{247}].
\end{array}$$

So

$$\begin{aligned}
p = [\mathbf{123}]\{&-[\mathbf{126}][\mathbf{127}][\mathbf{135}][\mathbf{245}][\mathbf{467}] + [\mathbf{125}][\mathbf{127}][\mathbf{136}][\mathbf{246}][\mathbf{457}] \\
&-[\mathbf{125}][\mathbf{126}][\mathbf{137}][\mathbf{247}][\mathbf{456}]\}.
\end{aligned}$$

Example 5.4 (From Example 7.3 in Section 7). Let

$$\begin{aligned} p \;=\; & -[\mathbf{125}][\mathbf{135}][\mathbf{145}][\mathbf{234}]^2 - [\mathbf{124}]^2[\mathbf{135}][\mathbf{235}][\mathbf{345}] \\ & + [\mathbf{125}][\mathbf{134}]^2[\mathbf{235}][\mathbf{245}] + [\mathbf{123}]^2[\mathbf{145}][\mathbf{245}][\mathbf{345}]. \end{aligned}$$

The lowest degree of pairwise terms with common bracket factors removed is $i = 4$. The points with their degrees are $\mathbf{1}^3, \mathbf{2}^2, \mathbf{3}^2, \mathbf{4}^2, \mathbf{5}^3$. The brackets with their total degrees are $[\mathbf{1j5}]^8$ for $2 \le j \le 4$, $[\mathbf{1jk}]^7$, $[\mathbf{jk5}]^7$ for $2 \le j < k \le 4$, and $[\mathbf{234}]^6$. Since there are five points, only explosions by 3-termed GP polynomials are possible.

We start from $[\mathbf{234}]$. Then $\mathcal{S} = \{\mathbf{1}, \mathbf{5}\}$. The first term p' is the only one that contains $[\mathbf{234}]$. All the other brackets in p' have total degree 8 and contain $\mathbf{1}, \mathbf{5}$. Start from any one, say $[\mathbf{125}]$. Then $p'' = -[\mathbf{135}][\mathbf{145}][\mathbf{234}]$.

The corresponding explosion is $[\mathbf{125}][\mathbf{234}] = [\mathbf{124}][\mathbf{235}] - [\mathbf{123}][\mathbf{245}]$. Its two terms are in the second and the last terms of p respectively. The two corresponding pairs are both contractible. The contractions are

$$[\mathbf{145}][\mathbf{234}] + [\mathbf{124}][\mathbf{345}] = [\mathbf{134}][\mathbf{245}], \quad [\mathbf{123}][\mathbf{345}] + [\mathbf{135}][\mathbf{234}] = [\mathbf{134}][\mathbf{235}].$$

So $p = [\mathbf{134}][\mathbf{235}][\mathbf{245}]([\mathbf{123}][\mathbf{145}] - [\mathbf{124}][\mathbf{135}] + [\mathbf{125}][\mathbf{134}])$. Another contraction changes it to zero.

Remark. Generally we only use level contractions and strong contractions to bracket polynomials whose different points have no incidence constraints, because the two transformations are relatively more complicated. This restriction is justified by theorem proving experiments. It is also possible to use Van der Waerden polynomials and successive contractions to reduce the number of terms. In our experiments, however, there is no need to do so.

6. Representation and elimination

In this section, we first analyse the nondegeneracy conditions in theorem proving, then propose elimination rules for a group of typical geometric constructions in projective geometry.

6.1. Geometric constructions and associated nondegeneracy conditions

In this paper, a free point on a line is always called a *semifree point*. A free point in a plane of the projective space is called a *semifree planar point*. By *free point* we mean exclusively free points in the projective plane or space. Conjugates, semifree points, semifree planar points, intersections of lines, intersections of lines and planes, and intersection points of planes, are called *incidence points*.

In incidence geometry, geometric constructions are generally a sequence of points. The *parents* of a point are its constructive points, the *children* of a point are the points whose parents include the point. The parent–child relationship defines a partial order among the points. For two comparable points, one is an *ascendant* of the other, and the opposite is *descendent*. In a Cayley expression p, the incidence points which are not ascendants of any other point in p, are called the *ends* of the expression.

When using elimination method to prove geometric theorems, there will be a set of inequality constraints (inequations) called *nondegeneracy conditions* (Kutzler and Stifter, 1986; Buchberger, 1988; Wu, 1994, 2000; Chou, 1988; Chou et al., 1995; Gao and Wang, 2000; Li, 2000; Wang, 2001; Zhang et al., 1995). They can be divided into two classes. The first class is the *given nondegeneracy conditions*. They are either explicitly or implicitly contained in the geometric constructions of a theorem. The following is a list of geometric constructions and the associated given nondegeneracy conditions.

Construction 1. $\mathbf{X}$ is a free point in the projective plane or space: no inequality constraint.

Construction 2. $\mathbf{X}$ is a semifree point on line $\mathbf{12}$: $\mathbf{1}$ and $\mathbf{2}$ are distinct, denoted by $\exists\mathbf{12}$.

Construction 3. $\mathbf{X}$ is a semifree planar point in plane $\mathbf{123}$: $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are not collinear, denoted by $\exists\mathbf{123}$.

Construction 4. $\mathbf{X}$ is the conjugate of a point $\mathbf{3}$ on line $\mathbf{12}$: $\exists\mathbf{12}$.

Construction 5. $\mathbf{X}$ is the intersection of two planar lines $\mathbf{12}, \mathbf{1'2'}$: $\exists\mathbf{12}$, $\exists\mathbf{1'2'}$, and points $\mathbf{1}, \mathbf{2}, \mathbf{1'}, \mathbf{2'}$ are not collinear.

Construction 6. $\mathbf{X}$ is the intersection of line $\mathbf{12}$ and plane $\mathbf{1'2'3'}$: $\exists\mathbf{12}$, $\exists\mathbf{1'2'3'}$, and points $\mathbf{1}, \mathbf{2}, \mathbf{1'}, \mathbf{2'}, \mathbf{3'}$ are not coplanar.

Construction 7. $\mathbf{X}$ is a semifree point on line $\mathbf{123} \cap \mathbf{1'2'3'}$, or $l = \mathbf{123} \cap \mathbf{1'2'3'}$: $\exists\mathbf{123}$, $\exists\mathbf{1'2'3'}$, and points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1'}, \mathbf{2'}, \mathbf{3'}$ are not coplanar; denoted by $\exists\mathbf{123} \cap \mathbf{1'2'3'}$.

Construction 8. $\mathbf{X}$ is the intersection of three planes $\mathbf{123}, \mathbf{1'2'3'}, \mathbf{1''2''3''}$: $\exists\mathbf{123}$, $\exists\mathbf{1'2'3'}$, $\exists\mathbf{1''2''3''}$, $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1'}, \mathbf{2'}, \mathbf{3'}$ are not coplanar, $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1''}, \mathbf{2''}, \mathbf{3''}$ are not coplanar, $\mathbf{1'}, \mathbf{2'}, \mathbf{3'}, \mathbf{1''}, \mathbf{2''}, \mathbf{3''}$ are not coplanar, and $\mathbf{123} \wedge \mathbf{1'2'3'} \wedge \mathbf{1''2''3''} \neq 0$.

The second class of nondegeneracy conditions is the *additional nondegeneracy conditions*. They are not needed by the geometric constructions, but are required by the proof of the theorem. In our algorithm, we always use division instead of pseudodivision. Since in homogeneous computing, common factors can always be removed, using division is just as efficient as using pseudodivision, with the exceptional benefit of obtaining additional nondegeneracy conditions directly from the denominators. Notice that the numerators of the common factors are NOT additional nondegeneracy conditions.

6.2. Free points and conjugates

A bracket is said to be *mute* if it only satisfies the relation **B2** in the definition of bracket algebra.

Elimination rule 1. Let $\mathbf{X}$ be a point. To eliminate $\mathbf{X}$ from a Cayley expression $p(\mathbf{X})$,

1. if $\mathbf{X}$ is a semifree point on line $\mathbf{12}$, then substitute the following formula into $p(\mathbf{X})$:

$$\mathbf{X} = [\mathbf{1X}]\mathbf{2} - [\mathbf{2X}]\mathbf{1}, \tag{6.1}$$

and set $[\mathbf{1X}]$, $[\mathbf{2X}]$ to be mute.

2. If $\mathbf{X}$ is the conjugate of a point $\mathbf{3}$ on line $\mathbf{12}$, then substitute the following formula into $p(\mathbf{X})$:

$$\mathbf{X} = [\mathbf{13}]\mathbf{2} + [\mathbf{23}]\mathbf{1}. \tag{6.2}$$

3. If $\mathbf{X}$ is a semifree planar point in plane $\mathbf{123}$, substitute the following formula into $p(\mathbf{X})$:

$$\mathbf{X} = [\mathbf{12X}]\mathbf{3} - [\mathbf{13X}]\mathbf{2} + [\mathbf{23X}]\mathbf{1}, \tag{6.3}$$

and set $[\mathbf{12X}]$, $[\mathbf{13X}]$, $[\mathbf{23X}]$ to be mute.

4. If $\mathbf{X}$ is a free point in the projective plane, first expand $p(\mathbf{X})$ into bracket polynomials. Order the bracket mates of $\mathbf{X}$ in $p(\mathbf{X})$ by their numbers of occurrences in the brackets containing $\mathbf{X}$. Let $\mathbf{1}, \mathbf{2}, \mathbf{3}$ be the first three bracket mates with maximal occurrences. Substitute into $p(\mathbf{X})$ the following formula:

$$\mathbf{X} = \frac{1}{[\mathbf{123}]}([\mathbf{12X}]\mathbf{3} - [\mathbf{13X}]\mathbf{2} + [\mathbf{23X}]\mathbf{1}). \tag{6.4}$$

Set $[\mathbf{12X}]$, $[\mathbf{13X}]$, $[\mathbf{23X}]$ to be mute.

5. If $\mathbf{X}$ is a free point in the projective space, first expand $p(\mathbf{X})$ into bracket polynomials. Order the bracket mates of $\mathbf{X}$ in $p(\mathbf{X})$ by their numbers of occurrences in the brackets containing $\mathbf{X}$. Let $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ be the first four bracket mates with maximal occurrences. Substitute into $p(\mathbf{X})$ the following formula

$$\mathbf{X} = \frac{1}{[\mathbf{1234}]}([\mathbf{123X}]\mathbf{4} - [\mathbf{124X}]\mathbf{3} + [\mathbf{134X}]\mathbf{2} - [\mathbf{234X}]\mathbf{1}). \tag{6.5}$$

Set $[\mathbf{123X}]$, $[\mathbf{124X}]$, $[\mathbf{134X}]$, $[\mathbf{234X}]$ to be mute.

Remark. (1) In (6.1)–(6.3), the coefficient of $\mathbf{X}$ is set to be 1 because it is always nonzero and $p(\mathbf{X})$ is homogeneous with respect to $\mathbf{X}$. (2) The above Cramer's rules can be taken as a *local* coordinate approach, in that each Cramer's rule introduces a coordinate system in a line, plane or space, which depends on the point $\mathbf{X}$ to be eliminated. In classical coordinate approach, however, the coordinate system is independent of the points, i.e. is *global*.

6.3. Intersections

Elimination rule 2. Assume that $\mathbf{X}$ is a point in the projective plane, and is the intersection of two lines $\mathbf{12}$, $\mathbf{34}$. To eliminate $\mathbf{X}$ from a Cayley expression $p(\mathbf{X})$ involving $\mathbf{X}$,

(a) if $\mathbf{X}$ is the only intersection of lines among the ends of $p(\mathbf{X})$, then for each scalar-valued component $q(\mathbf{X})$ of $p(\mathbf{X})$,

1. if $q(\mathbf{X}) = [\mathbf{56X}]$, replace $q(\mathbf{X})$ by $r = \mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56}$ if r has no factored expansion, or a factored expansion of r.
2. If $q(\mathbf{X}) = \mathbf{1}''\mathbf{X} \wedge \mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}'$, replace $q(\mathbf{X})$ by a factored/shortest expansion of $[\mathbf{1}''(\mathbf{12} \wedge \mathbf{34})(\mathbf{1}'\mathbf{2}' \wedge \mathbf{3}'\mathbf{4}')]$.

3. If $q(\mathbf{X}) = [\mathbf{X}(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]$, replace $q(\mathbf{X})$ by a factored/shortest expansion of $[(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]$.
4. In other cases, substitute $\mathbf{X} = \mathbf{12} \wedge \mathbf{34}$ into a factored/shortest expansion of $q(\mathbf{X})$.

(b) The intersections of lines which are ends of $p(\mathbf{X})$ can *be eliminated in a batch*: for each related bracket in $p(\mathbf{X})$, replace it by a factored/shortest expansion of the corresponding Cayley expression; for any other related Cayley expression in $p(\mathbf{X})$, first expand it into bracket polynomials, then replace the brackets by the corresponding factored/shortest expansions.

Remark. By our theorem proving experience, eliminating in a batch all intersections of lines which are ends of an expression can significantly speed up the proving procedure, with only minor decrease in the readability of the proof. However, eliminating other ends at the same time can often "blow up" the expression, and should be avoided.

Elimination rule 3. Let $\mathbf{X}$ be a point in the projective space.

1. If $\mathbf{X}$ is the intersection of two lines $\mathbf{12}$, $\mathbf{1'2'}$, to eliminate $\mathbf{X}$ from a bracket polynomial $p(\mathbf{X})$, (1) replace each bracket $[\mathbf{X1''2''3''}]$ in $p(\mathbf{X})$ by a factored/shortest expansion of $\mathbf{12U}_1 \wedge \mathbf{1'2'U}_1 \wedge \mathbf{1''2''3''}$, where $\mathbf{U}_1$ is a dummy vector, (2) eliminate $\mathbf{X}$ from each bracket $[\mathbf{X1''2''}]$ by Elimination rule 2.
2. If $\mathbf{X}$ is the intersection of line $\mathbf{12}$ and plane $\mathbf{1'2'3'}$, to eliminate $\mathbf{X}$ from a bracket polynomial $p(\mathbf{X})$, replace each bracket $[\mathbf{X1''2''3''}]$ in $p(\mathbf{X})$ by a factored/shortest expansion of $\mathbf{12} \wedge \mathbf{1'2'3'} \wedge \mathbf{1''2''3''}$.
3. If $\mathbf{X}$ is the intersection of three planes $\mathbf{123}, \mathbf{1'2'3'}, \mathbf{1''2''3''}$, to eliminate $\mathbf{X}$ from a bracket polynomial $p(\mathbf{X})$, replace each bracket $[\mathbf{X1'''2'''3'''}]$ in $p(\mathbf{X})$ by a factored/shortest expansion of $\mathbf{123} \wedge \mathbf{1'2'3'} \wedge \mathbf{1''2''3''} \wedge \mathbf{1'''2'''3'''}$.
4. If $\mathbf{X}$ is a semifree point on the line of intersection of planes $\mathbf{123}$, $\mathbf{1'2'3'}$, to eliminate $\mathbf{X}$ from a Cayley expression $p(\mathbf{X})$, substitute the following two expressions of $\mathbf{X}$ into every scalar-valued component $q(\mathbf{X})$ of $p(\mathbf{X})$ respectively, and select the shorter result in each $q(\mathbf{X})$:

$$\begin{aligned}\mathbf{X} &= ([\mathbf{X2}][\mathbf{31'2'3'}] - [\mathbf{X3}][\mathbf{21'2'3'}])\mathbf{1} - ([\mathbf{X1}][\mathbf{31'2'3'}] - [\mathbf{X3}][\mathbf{11'2'3'}])\mathbf{2} \\ &\quad + ([\mathbf{X1}][\mathbf{21'2'3'}] - [\mathbf{X2}][\mathbf{11'2'3'}])\mathbf{3} \\ &= ([\mathbf{X2'}][\mathbf{1233'}] - [\mathbf{X3'}][\mathbf{1232'}])\mathbf{1'} - ([\mathbf{X1'}][\mathbf{1233'}] - [\mathbf{X3'}][\mathbf{1231'}])\mathbf{2'} \\ &\quad + ([\mathbf{X1'}][\mathbf{1232'}] - [\mathbf{X2'}][\mathbf{1231'}])\mathbf{3'}. \end{aligned} \tag{6.6}$$

Proof of (6.6). The line of intersection contains the following six vectors:

$$\begin{aligned}&\mathbf{V}_1 = \mathbf{1'2'3'} \wedge \mathbf{23}, \quad \mathbf{V}_2 = \mathbf{1'2'3'} \wedge \mathbf{13}, \quad \mathbf{V}_3 = \mathbf{1'2'3'} \wedge \mathbf{12}, \\ &\mathbf{V}'_1 = \mathbf{123} \wedge \mathbf{2'3'}, \quad \mathbf{V}'_2 = \mathbf{123} \wedge \mathbf{1'3'}, \quad \mathbf{V}'_3 = \mathbf{123} \wedge \mathbf{1'2'}.\end{aligned}$$

Since $\mathbf{123} \wedge \mathbf{1'2'3'} \neq 0$, at least two of the six vectors, say $\mathbf{V}_1$ and $\mathbf{V}_2$, are not collinear. Substituting them into $[\mathbf{V}_1\mathbf{V}_2]\mathbf{X} = [\mathbf{V}_1\mathbf{X}]\mathbf{V}_2 - [\mathbf{V}_2\mathbf{X}]\mathbf{V}_1$, we get

$$\begin{aligned}&([\mathbf{31'2'3'}][\mathbf{12}] - [\mathbf{21'2'3'}][\mathbf{13}] + [\mathbf{11'2'3'}][\mathbf{23}])\mathbf{X}\\ &\quad = ([\mathbf{X2}][\mathbf{31'2'3'}] - [\mathbf{X3}][\mathbf{21'2'3'}])\mathbf{1} - ([\mathbf{X1}][\mathbf{31'2'3'}]\\ &\qquad -[\mathbf{X3}][\mathbf{11'2'3'}])\mathbf{2} + ([\mathbf{X1}][\mathbf{21'2'3'}] - [\mathbf{X2}][\mathbf{11'2'3'}])\mathbf{3}.\end{aligned} \tag{6.7}$$

Replacing $\mathbf{V}_1, \mathbf{V}_2$ by any other pair of vectors on the line, we obtain the same result (6.7). The representation becomes clearer when we use Cayley expressions. For two dummy vectors $\mathbf{U}_1, \mathbf{U}_2$,

$$\begin{aligned}\mathbf{U}_1\mathbf{U}_2\mathbf{X} \wedge \mathbf{123} \wedge \mathbf{1'2'3'} &= (\mathbf{U}_1\mathbf{U}_2 \wedge \mathbf{123} \wedge \mathbf{1'2'3'})\mathbf{X}\\ &= (\mathbf{U}_1\mathbf{U}_2\mathbf{X} \wedge \mathbf{123} \wedge \mathbf{2'3'})\mathbf{1'} - (\mathbf{U}_1\mathbf{U}_2\mathbf{X} \wedge \mathbf{123} \wedge \mathbf{1'3'})\mathbf{2'}\\ &\quad +(\mathbf{U}_1\mathbf{U}_2\mathbf{X} \wedge \mathbf{123} \wedge \mathbf{1'2'})\mathbf{3'}\\ &= -(\mathbf{U}_1\mathbf{U}_2\mathbf{X} \wedge \mathbf{1'2'3'} \wedge \mathbf{23})\mathbf{1} + (\mathbf{U}_1\mathbf{U}_2\mathbf{X} \wedge \mathbf{1'2'3'} \wedge \mathbf{13})\mathbf{2}\\ &\quad -(\mathbf{U}_1\mathbf{U}_2\mathbf{X} \wedge \mathbf{1'2'3'} \wedge \mathbf{12})\mathbf{3}.\end{aligned}$$

Expanding the r_I-typed wedge products, we get (6.7). Since $p(\mathbf{X})$ is homogeneous with respect to $\mathbf{X}$, the coefficient of $\mathbf{X}$, which is nonzero, can be removed. $\square$

7. Automated theorem proving

A salient feature of our theorem proving is *initial batch elimination*, a direct outcome of the Cayley expansion theory.

Algorithm: Initial batch elimination.

Input: A Cayley expression *conc*, and a construction sequence of points.

Output: *conc* after some eliminations and expansions, and the procedure to obtain it.

Procedure: Let $\mathcal{E}$ be the ends of *conc*.

(1) If *conc* is not composed of brackets and wedge products of type p_I, then expand it into bracket polynomials.
(2) In each related bracket or wedge product of *conc*, eliminate points in $\mathcal{E}$ at the same time by Cayley expansion and the elimination rules. If this is impossible for some wedge products, then expand the wedge products into bracket polynomials before the batch elimination.
(3) Contract and remove common factors of *conc*.

Below we present the main algorithm for short proof generation in incidence geometry, which is based on the techniques developed in the previous sections. The algorithm is implemented with Maple V.4, and has been tested by 35 theorems. Except for one theorem, the others all have 2-termed proofs, and the proofs often finish before one or several incidence points are eliminated. For the exceptional theorem (Example 7.3), it can be proved that there is no 2-termed proof, and the shortest proofs found by reconstructing the geometric configuration are 4-termed.

Algorithm: Short proof generation in incidence geometry.

Input: (1) A sequence of points together with their constructions, (2) a conclusion of the form $conc = 0$, where $conc$ is a Cayley expression.

Output: (1) A proving procedure, including Cayley expansions, eliminations, (strong, level) contractions, and removal of common factors; (2) additional nondegeneracy conditions.

Step 1 (Registration). Collect planes, lines and points. (1) A plane (or line) is a sequence composed of all points on it. The number of points is at least 4 (or 3). (2) A point is composed of the name and the construction.

Step 2 (Initial batch elimination). It is carried out to $conc$.

Step 3 (Elimination). Start from the highest-ordered element $\mathbf{X}$ in $conc$, do the following:

(1) If $conc = 0$ then go to Step 5, else if $conc$ has no incidence point, go to Step 4.
(2) Eliminate $\mathbf{X}$ from $conc$. Then do contraction and remove common factors.

Step 4 (Complete elimination). If there are wedge products in $conc$, then expand them into bracket polynomials and contract the result.While $conc \neq 0$ do the following. At the end of each step, do contraction and remove common factors.

(1) Do level contraction.
(2) Do strong contraction.
(3) Eliminate the last point of $conc$ in the construction sequence.

Step 5 (Additional nondegeneracy conditions). They are the denominators which are produced by Cramer's rules, and which are not cancelled after substitutions.

The completeness of the algorithm is guaranteed by the point-by-point elimination in Step 4. However, no theorem in our experiments needs to undergo the elimination of free points. All theorems except one finish by Step 3. The exception is Example 7.3, whose proof finishes after a strong contraction and a contraction without eliminating free points. For an equality of free points, the simplest proving method is to use Cramer's rule to introduce global coordinates, or to use the straightening algorithm to normalize it. For readable proving, we prefer the step-by-step transparent style.

7.1. Two-dimensional incidence geometry

Example 7.1 (Nehring's Theorem, See Example 1.1 in Section 1). We only show the procedure before eliminating point **A** in Example 1.1.

Rules	
	$[\mathbf{57C}]$
	$\overset{\mathbf{C}}{=} \mathbf{57} \wedge \mathbf{23} \wedge \mathbf{6B}$
$\boxed{\begin{aligned}&\mathbf{57}\wedge\mathbf{23}\wedge\mathbf{6B}\\&=-[\mathbf{235}]\mathbf{67}\wedge\mathbf{13}\wedge\mathbf{8A}-[\mathbf{237}]\mathbf{56}\wedge\mathbf{13}\wedge\mathbf{8A}\\&=-[\mathbf{235}][\mathbf{136}][\mathbf{78A}]-[\mathbf{237}][\mathbf{13A}][\mathbf{568}]\end{aligned}}$	$\overset{\mathbf{B}}{=} -[\mathbf{136}][\mathbf{235}][\mathbf{78A}] - [\mathbf{13A}][\mathbf{237}][\mathbf{568}].$

Additional nondegeneracy condition: none.

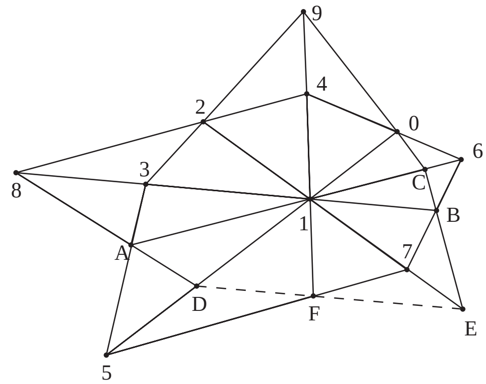

Fig. 2. Example 7.2.

Remark. Other 2-termed expansions of $\mathbf{57} \wedge \mathbf{23} \wedge \mathbf{6B} = [\mathbf{6}(\mathbf{13} \wedge \mathbf{8A})(\mathbf{57} \wedge \mathbf{23})]$ lead to much the same proofs.

Example 7.2. (Saam's Theorem, See Richter-Gebert, 1995, Example 6; Also Sturmfels, 1991, Proposition 2.1). Free points: **1, 2, 3, 4, 5, 6**.

Semifree point: **7** on **12**.

Intersections:

$$\begin{array}{lll} \mathbf{8} = \mathbf{13} \cap \mathbf{24}, & \mathbf{9} = \mathbf{23} \cap \mathbf{14}, & \mathbf{0} = \mathbf{15} \cap \mathbf{46}, \\ \mathbf{A} = \mathbf{35} \cap \mathbf{16}, & \mathbf{B} = \mathbf{13} \cap \mathbf{67}, & \mathbf{C} = \mathbf{16} \cap \mathbf{90}, \\ \mathbf{D} = \mathbf{15} \cap \mathbf{8A}, & \mathbf{E} = \mathbf{12} \cap \mathbf{BC}, & \mathbf{F} = \mathbf{57} \cap \mathbf{14}. \end{array}$$

Conclusion: **D, E, F** are collinear.

Proof.

Rules		[DEF]
$\mathbf{F}=[\mathbf{145}]\mathbf{7}-[\mathbf{147}]\mathbf{5}$ $[\mathbf{DE7}]=-[\mathbf{7BC}]\mathbf{15}\wedge\mathbf{8A}\wedge\mathbf{12}=-[\mathbf{7BC}][\mathbf{125}][\mathbf{18A}]$ $[\mathbf{DE5}]=[\mathbf{58A}]\mathbf{15}\wedge\mathbf{12}\wedge\mathbf{BC}=-[\mathbf{58A}][\mathbf{125}][\mathbf{1BC}]$	$\overset{\mathbf{D},\underline{\mathbf{E}},\mathbf{F}}{=}$	$\underbrace{[\mathbf{125}]}(-[\mathbf{145}][\mathbf{18A}][\mathbf{7BC}] + [\mathbf{147}][\mathbf{1BC}][\mathbf{58A}])$
$[\mathbf{18A}]=[\mathbf{135}][\mathbf{168}]$ $[\mathbf{58A}]=[\mathbf{156}][\mathbf{358}]$ $[\mathbf{7BC}]=-[\mathbf{137}]\mathbf{67}\wedge\mathbf{16}\wedge\mathbf{90}=[\mathbf{137}][\mathbf{167}][\mathbf{690}]$ $[\mathbf{1BC}]=[\mathbf{167}]\mathbf{13}\wedge\mathbf{16}\wedge\mathbf{90}=[\mathbf{136}][\mathbf{167}][\mathbf{190}]$	$\overset{\mathbf{A},\underline{\mathbf{B}},\mathbf{C}}{=}$	$\underbrace{[\mathbf{167}]}(-[\mathbf{135}][\mathbf{137}][\mathbf{145}][\mathbf{168}][\mathbf{690}] + [\mathbf{136}][\mathbf{147}][\mathbf{156}][\mathbf{190}][\mathbf{358}])$
$[\mathbf{168}]=-[\mathbf{124}][\mathbf{136}]$ $[\mathbf{358}]=[\mathbf{135}][\mathbf{234}]$ $[\mathbf{690}]=[\mathbf{156}]\mathbf{23}\wedge\mathbf{14}\wedge\mathbf{46}=[\mathbf{146}][\mathbf{156}][\mathbf{234}]$ $[\mathbf{190}]=-[\mathbf{123}]\mathbf{14}\wedge\mathbf{15}\wedge\mathbf{46}=-[\mathbf{123}][\mathbf{145}][\mathbf{146}]$	$\overset{\mathbf{8},\underline{\mathbf{9}},\mathbf{0}}{=}$	$\underbrace{[\mathbf{135}][\mathbf{136}][\mathbf{145}][\mathbf{146}][\mathbf{156}][\mathbf{234}]}([\mathbf{124}][\mathbf{137}] - [\mathbf{123}][\mathbf{147}])$
	$\overset{\text{contract}}{=}$	$0.$

Additional nondegeneracy condition: none.

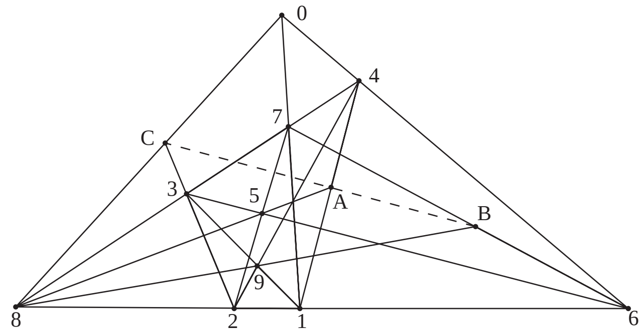

Fig. 3. Example 7.3.

Remark. $[\mathbf{DEF}] = [(\mathbf{15} \wedge \mathbf{8A})(\mathbf{12} \wedge \mathbf{BC})(\mathbf{57} \wedge \mathbf{14})]$ is the perspective pattern (3.43).

Example 7.3 (Leisening's Theorem, See Chou et al., 1994, Example 6.23). Let **126**, **347** be two lines. Let $\mathbf{5} = \mathbf{27} \cap \mathbf{36}$, $\mathbf{9} = \mathbf{24} \cap \mathbf{13}$, $\mathbf{0} = \mathbf{17} \cap \mathbf{46}$, and $\mathbf{8} = \mathbf{12} \cap \mathbf{34}$. Then the three intersections $\mathbf{85} \cap \mathbf{14}$, $\mathbf{89} \cap \mathbf{67}$, $\mathbf{80} \cap \mathbf{23}$ are collinear.

Free points: **3**, **6**, **8**.
Semifree points: **1**, **2** on **68**; **4**, **7** on **38**.
Intersections:
$\mathbf{5} = \mathbf{27} \cap \mathbf{36}$, $\mathbf{9} = \mathbf{24} \cap \mathbf{13}$, $\mathbf{0} = \mathbf{17} \cap \mathbf{46}$, $\mathbf{A} = \mathbf{58} \cap \mathbf{14}$, $\mathbf{B} = \mathbf{67} \cap \mathbf{89}$, $\mathbf{C} = \mathbf{23} \cap \mathbf{80}$.
Conclusion: **A**, **B**, **C** are collinear.
Analysis. By expanding

$$[\mathbf{ABC}] = [(\mathbf{58} \wedge \mathbf{14})(\mathbf{67} \wedge \mathbf{89})(\mathbf{23} \wedge \mathbf{80})], \tag{7.1}$$

in different ways, we find that after the intersections **5**, **9**, **0** are eliminated, the result contains either six or eight terms, so the shortest proofs are 6-termed. One reason is that there are too many semifree points in the construction. For this reason, below we reformulate the theorem without using semifree points.

Free points: **1**, **2**, **3**, **4**, **5**.
Intersections:

$$\begin{array}{llll} \mathbf{6} = \mathbf{12} \cap \mathbf{35}, & \mathbf{7} = \mathbf{34} \cap \mathbf{25}, & \mathbf{8} = \mathbf{12} \cap \mathbf{34}, & \mathbf{9} = \mathbf{24} \cap \mathbf{13}, \\ \mathbf{0} = \mathbf{17} \cap \mathbf{46}, & \mathbf{A} = \mathbf{58} \cap \mathbf{14}, & \mathbf{B} = \mathbf{67} \cap \mathbf{89}, & \mathbf{C} = \mathbf{23} \cap \mathbf{80}. \end{array}$$

Conclusion: **A**, **B**, **C** are collinear.

Proof.

Rules		[**ABC**]
$\mathbf{B}=[\mathbf{689}]\mathbf{7}-[\mathbf{789}]\mathbf{6}$, $\quad \mathbf{C}=[\mathbf{280}]\mathbf{3}-[\mathbf{380}]\mathbf{2}$ $[\mathbf{A73}]=-[\mathbf{148}][\mathbf{357}]$, $\quad [\mathbf{A72}]=-[\mathbf{145}][\mathbf{278}]$ $[\mathbf{A63}]=-[\mathbf{145}][\mathbf{368}]$, $\quad [\mathbf{A62}]=-[\mathbf{148}][\mathbf{256}]$	$\overset{\mathbf{A},\mathbf{B},\mathbf{C}}{=}$	$[\mathbf{145}][\mathbf{280}][\mathbf{368}][\mathbf{789}]$ $-[\mathbf{148}][\mathbf{280}][\mathbf{357}][\mathbf{689}]$ $-[\mathbf{148}][\mathbf{256}][\mathbf{380}][\mathbf{789}]$ $+[\mathbf{145}][\mathbf{278}][\mathbf{380}][\mathbf{689}]$

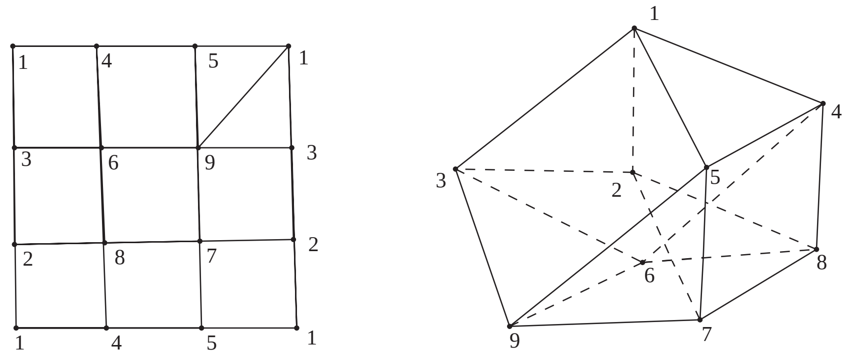

Fig. 4. Example 7.4.

$$\boxed{\begin{array}{l}
[\mathbf{368}]=[\mathbf{123}][\mathbf{346}]\\
[\mathbf{148}]=-[\mathbf{124}][\mathbf{134}]\\
[\mathbf{278}]=-[\mathbf{127}][\mathbf{234}]\\
[\mathbf{280}]=[\mathbf{234}]\mathbf{12}\wedge\mathbf{17}\wedge\mathbf{46}=[\mathbf{127}][\mathbf{146}][\mathbf{234}]\\
[\mathbf{380}]=-[\mathbf{123}]\mathbf{34}\wedge\mathbf{17}\wedge\mathbf{46}=[\mathbf{123}][\mathbf{134}][\mathbf{467}]\\
[\mathbf{689}]=[\mathbf{346}]\mathbf{12}\wedge\mathbf{24}\wedge\mathbf{13}=-[\mathbf{123}][\mathbf{124}][\mathbf{346}]\\
[\mathbf{789}]=-[\mathbf{127}]\mathbf{34}\wedge\mathbf{24}\wedge\mathbf{13}=[\mathbf{127}][\mathbf{134}][\mathbf{234}]
\end{array}}
\quad \overset{\mathbf{8},\underline{\mathbf{9}},\mathbf{0}}{=} \quad
\begin{array}{l}
\underbrace{[\mathbf{123}][\mathbf{127}][\mathbf{134}][\mathbf{234}]}\\
\{[\mathbf{127}][\mathbf{145}][\mathbf{146}][\mathbf{234}][\mathbf{346}]\\
-[\mathbf{124}]^2[\mathbf{146}][\mathbf{346}][\mathbf{357}]\\
+[\mathbf{124}][\mathbf{134}]^2[\mathbf{256}][\mathbf{467}]\\
+[\mathbf{123}][\mathbf{124}][\mathbf{145}][\mathbf{346}][\mathbf{467}]\}
\end{array}$$

$$\boxed{\begin{array}{l}
[\mathbf{127}]=-[\mathbf{125}][\mathbf{234}],\quad [\mathbf{146}]=-[\mathbf{124}][\mathbf{135}]\\
[\mathbf{346}]=-[\mathbf{123}][\mathbf{345}],\quad [\mathbf{357}]=[\mathbf{235}][\mathbf{345}]\\
[\mathbf{256}]=-[\mathbf{125}][\mathbf{235}],\\
[\mathbf{467}]=[\mathbf{245}]\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{34}=-[\mathbf{123}][\mathbf{245}][\mathbf{345}]
\end{array}}
\quad \overset{\underline{\mathbf{6}},\underline{\mathbf{7}}}{=} \quad
\begin{array}{l}
\underbrace{[\mathbf{123}][\mathbf{124}][\mathbf{345}]}\\
\{-[\mathbf{125}][\mathbf{135}][\mathbf{145}][\mathbf{234}]^2\\
\ -[\mathbf{124}]^2[\mathbf{135}][\mathbf{235}][\mathbf{345}]\\
\ +[\mathbf{125}][\mathbf{134}]^2[\mathbf{235}][\mathbf{245}]\\
\ +[\mathbf{123}]^2[\mathbf{145}][\mathbf{245}][\mathbf{345}]\}
\end{array}$$

$$\boxed{\begin{array}{l}
[\mathbf{125}][\mathbf{234}]=[\mathbf{124}][\mathbf{235}]-[\mathbf{123}][\mathbf{245}]\\
[\mathbf{123}][\mathbf{345}]+[\mathbf{135}][\mathbf{234}]=[\mathbf{134}][\mathbf{235}]\\
[\mathbf{145}][\mathbf{234}]+[\mathbf{124}][\mathbf{345}]=[\mathbf{134}][\mathbf{245}]
\end{array}}
\quad \overset{\text{strong}}{=} \quad
\begin{array}{l}
\underbrace{[\mathbf{134}][\mathbf{235}][\mathbf{245}]}([\mathbf{123}][\mathbf{145}]\\
-[\mathbf{124}][\mathbf{135}]+[\mathbf{125}][\mathbf{134}])
\end{array}$$

$$\overset{\text{contract}}{=} \quad 0.$$

Additional nondegeneracy condition: none.

Remark. The shortest expansions of (7.1) have four terms. However, not all of them lead to 4-termed proofs. There are only three ways leading to 4-termed proofs, which correspond to expanding two of the three pairs **58**, **67**, **23** hybridly in (7.1).

7.2. Three-dimensional incidence geometry

Example 7.4 (A Nonrealizable Torus, See Richter-Gebert, 1995, Example 13). Consider the configuration $\mathcal{C}$ with nine vertices, 19 edges and 10 facets, eight quadrangles and two triangles depicted in Fig. 4 (left). There does not exist a proper embedding of $\mathcal{C}$ into

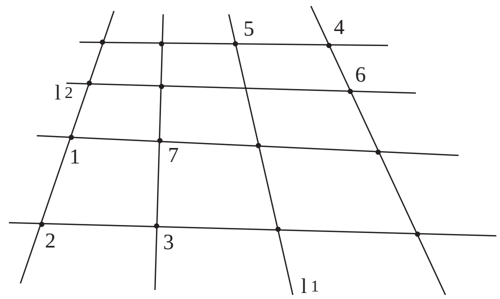

Fig. 5. Example 7.5.

Euclidean 3-space with all eight 4-sides facets as flat quadrangles, such that the two adjacent triangular faces **159** and **139** are not coplanar.

Free points: **1**, **2**, **3**, **4**, **5**.
Semifree points: **6** on **134**, **7** on **125**.
Intersections: $\mathbf{8} = \mathbf{124} \cap \mathbf{236} \cap \mathbf{457}$, $\mathbf{9} = \mathbf{237} \cap \mathbf{456} \cap \mathbf{678}$.
Conclusion: **1**, **3**, **5**, **9** are coplanar (Fig. 4, right).

Proof.

$$
\begin{array}{|l|cl}
\multicolumn{1}{c}{\text{Rules}} & & [\mathbf{1359}] \\
\hline
\mathbf{135} \wedge \mathbf{237} = [\mathbf{1235}]\mathbf{37} + [\mathbf{1357}]\mathbf{23} & & \\
\mathbf{37} \wedge \mathbf{456} \wedge \mathbf{678} = [\mathbf{4567}][\mathbf{3678}] & \overset{\mathbf{9}}{=} & [\mathbf{1235}][\mathbf{3678}][\mathbf{4567}] + [\mathbf{1357}][\mathbf{2367}][\mathbf{4568}] \\
\mathbf{23} \wedge \mathbf{456} \wedge \mathbf{678} = [\mathbf{4568}][\mathbf{2367}] & & \\
\hline
[\mathbf{3678}] = [\mathbf{2367}]\mathbf{36} \wedge \mathbf{124} \wedge \mathbf{457} & & \\
\qquad = -[\mathbf{2367}][\mathbf{1457}][\mathbf{2346}] & \overset{\mathbf{8}}{=} & \underbrace{[\mathbf{2367}][\mathbf{2346}][\mathbf{4567}]} \\
[\mathbf{4568}] = [\mathbf{4567}]\mathbf{45} \wedge \mathbf{124} \wedge \mathbf{236} & & ([\mathbf{1245}][\mathbf{1357}] - [\mathbf{1235}][\mathbf{1457}]) \\
\qquad = [\mathbf{4567}][\mathbf{1245}][\mathbf{2346}] & & \\
\hline
\multicolumn{1}{c}{} & \overset{\text{contract}}{=} & 0.
\end{array}
$$

Additional nondegeneracy condition: none.

Example 7.5. (Sixteen-Point Theorem, See Richter-Gebert, 1995, Examples 11 and 12). Let there be two groups of 3-D lines, each group containing four lines. When selecting one line from each group, there are 16 pairs of lines. If 15 pairs are coplanar ones, so is the 16th pair.

Free points: **1**, **2**, **3**, **4**.
Semifree points: **5** on **124**, **6** on **234**, **7** on $\mathbf{146} \cap \mathbf{345}$.
Intersections: $l_1 = \mathbf{157} \cap \mathbf{235}$, $l_2 = \mathbf{126} \cap \mathbf{367}$.

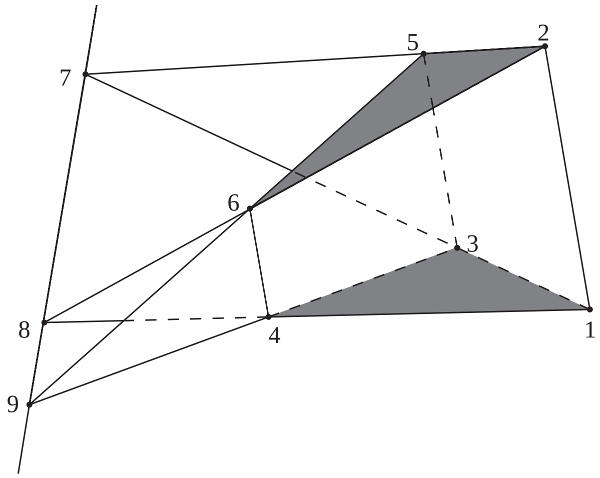

Fig. 6. Example 7.6.

Conclusion: l_1 and l_2 are coplanar.

Explanation of the construction: The eight lines are **12**, **17**, **23**, **37**, **45**, **46**, l_1 and l_2. The 16 pairs of lines are sequentially

$$\{\mathbf{12},\mathbf{17}\}\ \{\mathbf{12},\mathbf{23}\}\ \{\mathbf{23},\mathbf{37}\}\ \{\mathbf{37},\mathbf{17}\}\ \{\mathbf{45},\mathbf{46}\}\ \{\mathbf{45},l_1\}\ \{\mathbf{46},l_2\}\ \{\mathbf{12},\mathbf{45}\}$$
$$\{\mathbf{23},\mathbf{46}\}\ \{\mathbf{17},\mathbf{46}\}\ \{\mathbf{37},\mathbf{45}\}\ \{\mathbf{17},l_1\}\ \{\mathbf{23},l_1\}\ \{\mathbf{12},l_2\}\ \{\mathbf{37},l_2\}\ \{l_1,l_2\}$$

Proof.

Rules		$\mathbf{157}\wedge\mathbf{235}\wedge\mathbf{126}\wedge\mathbf{367}$
$\mathbf{157}\wedge\mathbf{235}=[\mathbf{1257}]\mathbf{35}-[\mathbf{1357}]\mathbf{25}$ $\mathbf{35}\wedge\mathbf{126}\wedge\mathbf{367}=-[\mathbf{1236}][\mathbf{3567}]$ $\mathbf{25}\wedge\mathbf{126}\wedge\mathbf{367}=-[\mathbf{1256}][\mathbf{2367}]$	$\overset{\text{expand}}{=}$	$[\mathbf{1256}][\mathbf{1357}][\mathbf{2367}]-[\mathbf{1236}][\mathbf{1257}][\mathbf{3567}]$
$\mathbf{7}=-([\mathbf{37}][\mathbf{1456}]+[\mathbf{57}][\mathbf{1346}])\mathbf{4}$ $+[\mathbf{47}][\mathbf{1456}]\mathbf{3}+[\mathbf{47}][\mathbf{1346}]\mathbf{5}$	$\overset{\mathbf{7}}{=}$	$\underbrace{[\mathbf{47}]([\mathbf{37}][\mathbf{1456}]+[\mathbf{57}][\mathbf{1346}])}(-[\mathbf{1256}][\mathbf{1345}][\mathbf{1346}][\mathbf{2356}]-[\mathbf{1235}][\mathbf{1236}][\mathbf{1456}][\mathbf{3456}])$
$\mathbf{6}=[\mathbf{346}]\mathbf{2}-[\mathbf{246}]\mathbf{3}+[\mathbf{236}]\mathbf{4}$	$\overset{\mathbf{6}}{=}$	$0.$

Additional nondegeneracy condition: none.

Example 7.6 (Solid Desargues Theorem, See Brannan et al., 1998, p. 101). If two triangles in the projective space correspond in such a way that the lines joining corresponding vertices are concurrent, then the three intersections of the corresponding pairs of sides are collinear.

Free points: **1**, **2**, **3**, **4**.
Semifree point: **5** on **123**, **6** on $\mathbf{124}\cap\mathbf{345}$.
Intersections: $\mathbf{7}=\mathbf{13}\cap\mathbf{25}$, $\mathbf{8}=\mathbf{14}\cap\mathbf{26}$, $\mathbf{9}=\mathbf{56}\cap\mathbf{34}$.
Conclusion: **7**, **8**, **9** are collinear.

Proof.

Rules
$[(\mathbf{13}\wedge\mathbf{25})(\mathbf{14}\wedge\mathbf{26})(\mathbf{34}\wedge\mathbf{56})] = -[\mathbf{134}][\mathbf{256}]\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{46}$
$\mathbf{6}=([\mathbf{26}][\mathbf{1345}]-[\mathbf{16}][\mathbf{2345}])\mathbf{4} - [\mathbf{16}][\mathbf{1345}]\mathbf{2}+[\mathbf{16}][\mathbf{2345}]\mathbf{1}$

$$
\begin{aligned}
& [\mathbf{789}] \\
\overset{\mathbf{7,8,9}}{=} \; & -\underbrace{[\mathbf{134}][\mathbf{256}]}\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{46} \\
\overset{\mathbf{6}}{=} \; & \underbrace{[\mathbf{16}][\mathbf{124}]}([\mathbf{135}][\mathbf{2345}]-[\mathbf{235}][\mathbf{1345}]) \\
\overset{\text{contract}}{=} \; & 0.
\end{aligned}
$$

Additional nondegeneracy condition: none.

Remark. The projective space can be n-dimensional for any $n > 1$. The algebraic identity (Cayley expansion)

$$[(\mathbf{13}\wedge\mathbf{25})(\mathbf{14}\wedge\mathbf{26})(\mathbf{34}\wedge\mathbf{56})] = -[\mathbf{134}][\mathbf{256}]\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{46} \tag{7.2}$$

holds in nD projective space as long as the brackets and wedge products are understood to be Mourrain ones, i.e.

$$
\begin{array}{lll}
[\mathbf{134}] & \longrightarrow & [\mathbf{134U}_1\mathbf{U}_2\ldots\mathbf{U}_{n-2}], \\
\mathbf{13}\wedge\mathbf{25} & \longrightarrow & \mathbf{13}\wedge\mathbf{25U}_1\mathbf{U}_2\ldots\mathbf{U}_{n-2}, \\
\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{46} & \longrightarrow & \mathbf{12}\wedge\mathbf{35U}_1\mathbf{U}_2\ldots\mathbf{U}_{n-2}\wedge\mathbf{46U}_1\mathbf{U}_2\ldots\mathbf{U}_{n-2}, \text{ etc.}
\end{array}
$$

The Desargues theorem in nD projective geometry is just (7.2), or equivalently, let $U = \mathbf{U}_1\mathbf{U}_2\ldots\mathbf{U}_{n-2}$,

$$[(\mathbf{13}\wedge\mathbf{25}U)(\mathbf{14}\wedge\mathbf{26}U)(\mathbf{34}\wedge\mathbf{56}U)U] = -[\mathbf{134}U][\mathbf{256}U]\mathbf{12}\wedge\mathbf{35}U\wedge\mathbf{46}U. \tag{7.3}$$

From this aspect, the proof of the solid Desargues Theorem is not necessary.

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